Chain-oscillations and polygons in physics

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Abstract

In this work, chain-oscillations of liquid capillary jets are introduced, and studied theoretically and experimentally. The normal modes of vibration of capillary liquids are addressed in this context, based on earlier work of Lord Rayleigh [1] and Niels Bohr [2]. This includes an experimental confirmation of Bohr’s non-linear corrections on the frequency of oscillation of a cylindrical capillary jet. The normal modes of a capillary drop (and jet) are re-interpreted and new solutions are found for the movement about the center of mass. Additionally, we discuss the origin of polygonal patterns (or “star-shapes”) that emerge in various domains of physics and over a multitude of length-scales. We show how some of those phenomena correspond indeed to similar dynamics, leading to similar shapes. Many of these examples can be attributed to the scale-invariant solutions of the Laplace-Equation.

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0.1 Foreword: The history and idea behind this Master’s Project

This all started with an observation I made during my Bachelor’s Degree in Physics Engineering at the university Instituto Superior Técnico (IST) in Lisbon, Portugal: after a lecture on oscillations & waves, during which we discussed various universal traits and properties of oscillatory behaviour, I noticed how the water stream coming from a tap (under steady flow) exhibited fairly peculiar dynamics. I gazed at it until I was sure to see oscillatory motion: a well defined wavelength and standard exponential damping behaviour. The water stream appeared to have a sort of helical twist-like shape or, at least, a very unique geometry. Since then, I started seeing the same oscillations in situations other than a tap. When a liquid is poured (for instance juice, wine or milk) these oscillations are very often present. It is also common to observe this effect during urination, although both in urination and pouring, the shape is usually not steady, as it is difficult to maintain the same flow conditions over time.

Later, in Amsterdam, during the first year of my Master’s Degree, I took the course "Hydrodynamics (Microfluidics)", in which we had to present our understanding of a physical phenomenon, in a 10 min talk at the end of the course, either through theoretical or numerical work. With the understanding gained during that course, I took that small effect I had observed, and explained qualitatively (through dimensional analysis and “back of the envelope calculations”) what was happening. The department in Amsterdam was working on the description and understanding of spraying, and as such, a seemingly unexplored oscillation of a jet posed a very promising research project. The presentation I did at that point, ended up becoming my Master Thesis, which in turn lead me to this work’s conclusions. The thesis was supposed to be the description of these oscillations in liquid jets, and their implications on the jet’s stability. The title was originally:

“Chain-oscillations in liquids and surface instabilities”.

My hypothesis was that these oscillations were induced by the asymmetry of the curvature of the surface and governed by the Young-Laplace Equation. More concretely, I postulated that whenever a liquid comes out of an orifice that is some stretched ellipsoid, surface tension on the free surface would act on a slice of the passing liquid, making it oscillate into a circle, overshooting into a 90° turned ellipsoid, then back to a circle once more; and then the whole process repeats. At steady initial conditions, this oscillation produces a sort of stationary chain. Falling at approximately constant speed in the −z direction, it displays alternating elongations in the x and y axes and circular cross-sections at the nodal points. Hence the name “Chain-oscillations”.
It seemed reasonable as a first approximation, to start off assuming that oval shape to be an ellipse. To test this elasto-capillary hypothesis, an experiment was designed, in which water flows through a jet stabilizer (to make the flow as laminar as possible) and out of a “perfectly” ellipse-shaped orifice. As one can observe in Figure 0.1, the jet does produce “chain-oscillations”, which suggests that, indeed, a non-circular initial cross-section of the liquid jet is enough to induce these dynamics. Even though that might not imply that this is the only way this effect manifests (which it is not), it certainly made it very likely that the hypothesis was right.

I also postulated, that if a circular jet had some vorticity (or angular velocity), the centrifugal force should be able to elongate it in the same fashion, and the jet would break the symmetry and swirl in a helix, with the winding frequency and the elongation proportional to the angular speed. In principle one should be able to excite both effects, creating a “chain-oscillating” spiral.

Shortly after finding the shapes of the normal modes of a 2D drop (which can be thought of as the cross-section of chain-oscillations), we were presented with some references, in which these modes were already studied dynamically as waves and oscillations in 2D (polar coordinates) and 3D (cylindrical and spherical coordinates). The works in question are [1], by Lord Rayleigh and [2], by Niels Bohr. At a later stage of my Master’s dissertation, I started finding the shapes of the normal modes of drops — or at least remarkably similar shapes — in many different systems across physics. To my surprise, these shapes seemed to manifest in all the (known) phases of matter, at different length-scales and across different domains of physics. As a theoretical physics student and enthusiast, it is needless to say, that the focus of this work shifted to trying to wrap my head around why and how this would be possible. As such, this work is presented in 3 parts:

(I) The study of chain-oscillations,

(II) How these relate to the normal modes of a (capillary) drop,

(III) How one can use these ideas to understand the emergence of some polygonal patterns in physics.
Part I: Chain-oscillations

1.1 Introduction

"Chain-oscillations" are a very peculiar dynamical effect, arising in liquid jets. They manifest best in liquid interfaces with a high surface tension between the two phases and a low viscosity in both. A good example thereof is a water jet surrounded by air, which is why it should not come as a surprise that this effect can be observed in quite a lot of everyday effects. We found that this effect can easily be confused with the Rayleigh-Plateau instability, or with a liquid helix. However, this thesis shows that these are distinct phenomena, although they are indeed related.

As shall be seen, chain-oscillations are a stable mode of vibration of a capillary jet, contrary to the unstable modes that correspond to the Rayleigh-Plateau instability. In addition, a liquid helix can look similar to chain-oscillations, but it does not produce the same shape. The shape of a chain is not a common geometry to be found in nature or physics. However, as shall be seen, this geometry emerges very naturally as a perturbation of a cylindrical object. In the remainder of this document we shall get to the heart of the dynamics of this shape and what it represents phenomenologically. But first, we invite the reader to appreciate the indisputable beauty, that surrounds this phenomenon:

![Figure 1.1: Pictures of chain-oscillations in water.](image)

1.2 Phenomenological description

This is a capillary phenomenon, in other words, these oscillations manifest on a liquid jet due to surface tension. As a jet is forced to flow in one direction ($-z$), its surface area cannot be minimized in that dimension. But in the radial plane ($x, y$), surface tension is free to act. Therefore, whenever a jet’s cross-section is not circular, it evolves as it flows down the jet. At steady initial conditions, the evolution of each passing "slice" of the liquid jet is identical, which produces a stationary geometry: the evolution of each portion of the liquid is determined by the shape of the orifice, the properties of the liquid and the jetting conditions. Therefore, when all these are kept constant, the surface of the capillary jet appears to be stationary, even though there is a continuous flow of liquid. Consequently, for a given orifice - with any particular shape or geometry - a capillary jet can assume a seemingly static form, when the flow-rate is kept steady.

Chain-oscillations are observed when a liquid flows out of a stretched, non-circular orifice. For instance, when water is expelled from an oval opening, surface tension acts on the oval cross-section of the produced jet. In doing so, the cross-section is pulled back into a circular form, which is the equilibrium shape of the jet’s cross-section. However, in the case of low damping, the liquid builds up momentum as it deforms back into a
circle, making it overshoot the circular shape: as the circular shape is approached, the initially stretched axis is compressed further - and the initially compressed axis stretched - which results in an oval cross-section again, but rotated 90° to the original orientation. As the process repeats - again rotating 90° - the initial orientation of the oval cross-section is reached again. This completes one oscillation cycle, as depicted in Figure 1.2.

Figure 1.2: Oscillation cycle of the cross-section of "pure" chain-oscillations.

The cross-section of the jet repeats this cycle as it flows along the jet-axis, producing the chain-like geometry in Figure 1.3. Let it be noted, that the shapes displayed in Figure 1.2 correspond to the cross-section of an "ideal" (or "pure") chain-oscillation in a capillary jet at large wavelengths. This relates to the theoretical description of the phenomenon, which will be addressed shortly. In Part (II) of this document, this shall be discussed further, where we show how the shapes of Figure 1.2 were obtained. To get a clear notion of their shape, chain-oscillations should be observed from different perspectives. Their geometry becomes apparent as one rotates the liquid chain around its cylindrical axis, observing the effect from different angles:

Figure 1.3: Different perspectives of chain-oscillations in water (and air), respectively: 0°, 45°, 90°

Notice that the chain-figure seems completely symmetric, when observed at 45° to its elongations, as can be observed in Figure 1.3. As shown in Figure 1.4, this is also the case for a standard metal chain; a seemingly small detail, which will turn out to be quite important for our experiments, when measuring the shape.
In theory, these oscillations should repeat indefinitely until they damp out. Thus, the oval elongations should slowly reduce in amplitude until the jet has a perfectly cylindrical shape, that is, a circular cross-section. In practice, however, we find that the liquid jet breaks up a few centimeters after its formation, before chain-oscillations damp out. This is due to the Rayleigh-Plateau instability, whose perturbations grow superimposed to these oscillations, breaking up the liquid jet anyway far downstream. This can be observed in the two pictures on the right of Figure 1.1.

Common everyday examples of chain-oscillations are some faucets, when their opening is oval. But the phenomenon can be also observed while pouring a liquid out of a small container (e.g. a cup, bottle, cooking vessels, ...) and during urination - although in the latter two examples it is difficult to maintain steady jetting conditions.

1.3 Underlying theory

A theoretical description of the vibrations of a capillary jet was developed by Lord Rayleigh [1], and later improved by Niels Bohr [2]. A brief summary of their work is presented in Appendix A. The accurate mathematical description of these vibrations can become very intricate and cumbersome to understand. Their theoretical description is the subject of Part (II) of this document, and therefore in this section we will discuss the underlying theory in its simplest form. In this way, we hope to bring some understanding to the phenomenon first, and to introduce the relevant quantities needed for the experiments that follow, before exploring the theory.

We can think of an ideal cylindrical jet as having a fixed circular cross-section with a constant radius $a$. When perturbed, its shape can be described by a boundary shape $R(\phi, z)$ in cylindrical coordinates. Up to small deformations, and taking only the effect of surface tension into account, a self consistent linear theory predicts that:

$$R(\phi, z) = a + \sum_{n} b_n \cos(n\phi) \cos(kz) \quad \forall n \in \mathbb{N}_0,$$  

(1.1)

where $k = \frac{2\pi}{\lambda}$ is the wave-vector for an associated wavelength $\lambda$. This presents a normal mode expansion for the perturbations of a cylindrical jet: any arbitrarily deformed boundary can be thought of as being the linear superposition (sum) of their respective normal mode components with amplitude $b_n$. Conversely, one can in theory produce any boundary shape by adding up different components of "pure modes". A pure mode $n$ corresponds to a perturbation of exclusively one particular perturbation, $R_n(\phi, z) = a + b_n \cos(n\phi) \cos(kz)$. This normal mode expansion is valid in the regime of Rayleigh's linear theory (small amplitudes $b_n$). However, in Part (II) we will argue how this can be generalized past this regime, when going deeper into the theoretical description.
Because surface tension expresses an energy cost per surface area, a deformed boundary of the jet has an associated cost in potential energy (surface energy). For a pure mode at a given amplitude, that energy cost $E_n^{\text{surf}}$ is proportional to:

$$E_n^{\text{surf}} \propto \frac{\gamma}{a} (k^2 a^2 + n^2 - 1) b_n^2 ,$$  

(1.2)

where $\gamma$ represents the surface tension. One can see how these are typical normal modes, as the energy per elongation for a given mode increases with mode order $n$, and the functions describing them are orthogonal, $\forall n_1 \neq n_2$. We can also see that the system can lower its energy when $n = 0$ and $|ka| < 1$, which corresponds to the Rayleigh-Plateau instability. There are thus unstable circularly-symmetric modes ($n = 0$), which grow spontaneously on the jet’s surface, breaking up the jet into drops. In contrast, we see that the non-circular modes $n \geq 1$ are stable. If the jetting speed $v_j$ is uniform, the flow is laminar and the motion can be thought of as the 2D oscillation of the cross-section of the jet as it flows along the jet axis. The associated frequency can thus be thought of as the frequency of oscillation of a slice of the liquid jet. In this laminar regime $\omega_n t = k z$, so that $v_z = \frac{\omega_n a}{k} = \frac{\Delta \gamma}{\rho a^2}$. The jet’s cross-section behaves exactly like an oscillating 2D drop (2D drop) in the limit of large wavelengths, that is $|ka| \to 0$. In this limit, the angular frequency of oscillation $\omega_n$, is given by:

$$\omega_n = 2\pi f_n = \sqrt{\frac{(n^3 - n)\gamma}{\rho a^3}} ,$$  

(1.3)

where $\rho$ is the density of the liquid and $f_n$ denotes the frequency of a pure mode $n$. This formula can then be thought of as suffering corrections for additional effects, such as the viscosity, the surrounding air, gravity, small wavelengths ($|ka|$), and large amplitudes $b_n$.

Chain-oscillations should be understood as one of the non-circular modes. In particular, given the oval symmetry of their cross-section, it can be argued from the boundary shape $R(\phi, z)$, that chain-oscillations should correspond to the expression of the mode $n = 2$. This is because the term proportional to $\cos (2\phi)$ has exactly 2 maxima and 2 minima around the circle, $\phi \in [0, 2\pi]$, spaced apart 90° from each other. As such, it is the sole term in (1.1) whose effect is to stretch the circle along one axis (for instance along the x-axis) and to compress it perpendicular to it (along the y-axis). In other words, $n = 2$ is the only term capable of producing an elongated oval shape. Moreover, note that the resulting oval shape is symmetric, since $R_2(\phi, z) = R_2(\phi \pm \pi, z)$. This becomes apparent by drawing the boundaries of the modes at a fixed height, which corresponds to the cross-section of a jet that displays a pure mode $n$:

$R_n(\phi, z = 0) = R_n(\phi) = a + b_n \cos (n\phi)$. In doing so, we obtain the shapes displayed in Figure 1.5:

---

Figure 1.5: Cross-section $R_n(\phi)$ of modes $n = 2$ to $n = 6$, with $R_n(\phi) = a + b_n \cos (n\phi)$.

As is plain to see in Figure 1.5, $n = 2$ is indeed the only mode capable of producing an elongated shape. The remaining modes $n \geq 3$ produce polygon-like shapes with $n$ maxima and $n$ minima distributed symmetrically around the circle. Note that for each pure mode, the cross-section of the jet will appear to rotate $\frac{180^\circ}{n}$ after each half-oscillation $\frac{1}{2}$, as $\cos (kz)$ changes sign. In other words, for $n \geq 2$ we have:

$$\left\{ \begin{array}{l}
\cos \left( k \left[ z \pm \frac{\lambda}{2} \right] \right) = -\cos (kz) \\
\cos \left( n \left[ \phi \pm \frac{180^\circ}{n} \right] \right) = -\cos (n\phi)
\end{array} \right\} \implies R_n(\phi, z) = R_n \left( \phi \pm \frac{180^\circ}{n}, z \pm \frac{\lambda}{2} \right) .$$  

(1.4)
Given their chain-like geometry, the “links” of chain-oscillations are to be at 90° to each other, as can be observed in Figure 1.3. Their “links” are defined in analogy to the links of a standard metal chain, as displayed in Figure 1.4. Since the wavelength \( \lambda \) is the distance over which the structure of chain-oscillations repeats itself, and considering that exactly 2 links fit in one wavelength, the link-length of chain-oscillations corresponds to half a wavelength, \( \frac{\lambda}{2} \). As we just argued above, each mode appears to rotate \( \frac{180°}{n} \) after each half-wavelength. Again, this suggests that chain-oscillations correspond to the expression of mode \( n = 2 \), because:

\[
90° = \frac{180°}{n} \quad \Rightarrow \quad n = 2
\]

(1.5)

### 1.4 Producing chain-oscillations

Chain-oscillations can easily be produced by forcing a steady flow through an elliptical orifice, which is how we generated them. In fact, later we discovered that this is how they were studied in previous investigations too, [1, 2, 3, 4]. Given that the deformed liquid jet is still subject to breaking up, we found that the effect was most stable for large, smooth jets. This suggests, that chain-oscillations are most stable in a laminar flow regime. For the jet to be as smooth as possible, our elliptical orifices were manufactured with a precision wire cutter, and the flow was laminarized before the outlet using a "smooth jet setup". This way we were able to produce very smooth jets, with a breakup length of the order of 1 meter.

![Image of chain-oscillations and orifice](image.png)

Figure 1.6: (a) Laminar nozzle for our “smooth jet setup” with the elliptical orifice 8. (b) The produced liquid chain with the definition of the wavelength \( \lambda \) and boundary-shape function \( R(\phi, z) \).

Our "smooth jet setup" comprises the use of a "laminar nozzle" just before the outlet at the orifice, as illustrated in Figure 1.6. A more detailed description of the experimental setup can be found in Appendix B, as well as the specifications of the used orifices. In our studies, we made use of a total of 13 distinct elliptical orifices. Since we are using a fixed liquid (water) and a rigid orifice, keeping stable jetting conditions amounts to maintaining a fixed flow-rate \( Q \), which is our control parameter. As we set \( Q \) to a fixed value, we found that the chain-figure would indeed become stationary, after a brief transitory period of about 1 second; this should correspond to the time it took for the pressure distribution to even out along the setup. An example of the produced jets for a varying control-parameter (the flow-rate \( Q \)) is presented below in Figure 1.7.
As can be seen in Figure 1.7, the wavelength of chain-oscillations increases with increasing flow-rate. Notice further how the surface of the liquid chain becomes increasingly irregular for increasing flow-rates.

1.5 Experiment: The frequency of chain-oscillations

Let us start by discussing our quantitative experimental observations, before exploring the theory further in Part (II) of this document. In our experiments, we focus on measuring the frequency of chain-oscillations and comparing it to the theoretical predictions. Lord Rayleigh presents the simplest prediction, which can then be corrected to account for additional effects, such as the viscosity, the surrounding air, gravity, small wavelengths, and large amplitudes. In showing how to determine the frequency accurately, we hope to attain a deeper understanding of the phenomenon.

1.5.1 Introducing the experiment

At a given flow-rate $Q$, the chain-figure is stationary, which through volume conservation implies a constant cross-section $A = \pi a^2$ at every height. Consequently, we can determine the frequency of oscillation experimentally by taking a picture of the stationary chain-figure and measuring the wavelength $\lambda$ directly from the image:

$$\begin{cases} Q = Av_z \\ f = \frac{v_z}{\lambda} \quad \Rightarrow \quad f : = f_\lambda = \frac{Q}{\pi a^2 \lambda} \end{cases} .$$  \hspace{1cm} (1.6)

In reality the jet will experience thinning due to the effect of gravity. In practice we found that this was negligible for our experimental conditions. We shall come back to this detail shortly.
We will thus refer to the experimental values of the chain-oscillation’s frequency as \( f_\lambda \), since they are obtained through the measurement of the wavelength, as shown in Figure 1.6 and Figure 1.7. Note that above we defined the cross-section of the chain-oscillating jet to be given by \( A = \pi a^2 \). The next step is to use a theoretical prediction and to investigate whether the theoretical and experimental values of the frequency coincide. Surmising that chain-oscillations correspond to the mode \( n = 2 \), we see from (1.3) that Rayleigh’s linear theory estimates their frequency as \( f = \frac{1}{2\pi} \sqrt{\frac{\gamma}{\rho a^3}} \). Because the jet originates from an orifice with a given area \( A_{\text{orifice}} \), the length-scale \( a \) is often taken as the radial length-scale of the orifice [3, 4]. Simply put, if we define an effective length-scale associated to the orifice, for instance \( A_{\text{orifice}} = \pi R_0^2 \), then the length-scale \( a \) can easily be mistaken with the length-scale of the orifice \( R_0 \).

However, an important observation from our experiments is that this assumption is not justified. The length-scale \( a \) should represent the size of the jet, not the size of the orifice. And the cross-section of the jet depends on the flow-rate \( Q \) and is significantly smaller than the constant area of the orifice. In fact, it is on average less than half the size, for the conditions studied in our experiments. This is shown in Figure 1.8 for one specific orifice, yet this observation holds for all the orifices we used.

\[ \begin{align*}
\text{Figure 1.8: Areas } A \ [\text{cm}^2] \text{ as a function of flow-rate } Q \ [\text{L/min}] \text{ for orifice 5. The constant line is the area of the elliptical orifice } A_{\text{orifice}} = \pi R_0^2 \text{ = constant. The dots represent the area of the cross-section of the chain-oscillating jet, } A_{\text{jet}} = A_{\text{jet}}(Q) = \pi a^2(Q). \text{ These were obtained through indirect measurements of the radius of the chain-figure, } a(Q), \text{ as a function of the flow-rate.}
\end{align*} \]

As is plain to see from Figure 1.8, the radial scale of the chain-figure \( a \) depends on the flow-rate \( Q \) and does not match the constant radial scale of the orifice \( R_0 \). The reason why these length-scales differ (and the respective cross-sections) merits some discussion. However, let us focus on our experiments first and we shall come back to this issue in a later section. Clearly, it is not accurate to use the length-scale of the orifice, \( R_0 \), for the determination of the chain-oscillation’s frequency. Instead, the radial scale of the liquid chain, \( a \), should be used, which has to be measured from the chain-figure. To do so, we first define the boundary-shape of the chain-figure \( R(\phi, z) \), so that its cross-section is \( A = \pi a^2 \):

\[ R(\phi, z) \simeq a + b \cos(2\phi) \cos(kz) \quad . \]

One should think of \( a \) as the radius of the cross-section of the jet, and \( b \) as the amplitude of the chain-oscillations. The parameters \( a \) and \( b \) describing the liquid chain can then be determined by measuring the largest (and smallest) radius that the chain-figure displays:

\[ \begin{align*}
\begin{cases}
\frac{1}{2}(R_{\text{max}} - R_{\text{min}}) = b \\
\frac{1}{2}(R_{\text{max}} + R_{\text{min}}) \simeq a 
\end{cases} \quad . \tag{1.8}
\end{align*} \]

\( R_{\text{max/min}} \) are measured from the same picture as the wavelength, needed for (1.6). Note that for this measurement to work, the figure should be measured at \( 0^\circ \) (or \( 90^\circ \)), following the convention of Figure 1.3. Moreover, equations (1.7) and (1.8) represent an oversimplified version of the method we used, presented here for simplicity. The details of the more elaborate calculations can be found in Appendix B. There, one can also find the
profile of the measured parameters \(a(Q)\) and \(b(Q)\) as a function of flow-rate for all our orifices. An interesting observation is that both these parameters depend strongly on the flow-rate, and increase with increasing flow-rate. As a matter of fact, so does the adimensional amplitude of the liquid chain-figure, \(\varepsilon \equiv \frac{b}{a} = \varepsilon(Q)\). There, one can also compare the length-scale \(R_0\) with the profiles of \(a(Q)\), to verify that these do indeed differ for all used orifices.

Let us now briefly summarize the methods of our experiments. For a given orifice, the flow-rate is varied. Once the chain-figure is stationary, the flow-rate is measured and a picture is taken at 0°, following the convention of Figure 1.3. From the picture, we measure the wavelength and the radial scales \(R_{\text{max}}\) and \(R_{\text{min}}\). In this manner, we acquire the wavelength \(\lambda\) and the parameters of the chain-figure \(a\) and \(b\) for each value of the flow-rate \(Q\). Using (1.6) we are able to experimentally determine the frequency \(f_\lambda\) as a function of \(Q\). We can then compare this experimental measurement with theoretical predictions, which are evaluated from the same measurements of the parameters \(a\) and \(b\). Note that a picture has to be taken for each data point \((Q)\), because all the measured quantities depend on the flow rate.

### 1.5.2 Results

As mentioned previously, the simplest theoretical estimate corresponds to (1.3), and was first derived by Rayleigh. It is obtained by only considering the effect of surface tension, and linearizing the corresponding equations. In other words, only linear terms in amplitude \(b\) are kept, while higher order terms are assumed negligible and discarded. This comprises a 1st order model, whose validity is restricted to small values of the amplitude, meaning \(\varepsilon = \frac{b}{a} \ll 1\). More sophisticated theoretical predictions can be obtained by correcting (1.3) to account for additional effects, such as the viscosity, the surrounding air, gravity, small wavelengths, and large amplitudes. This presents quite a large variety of different theoretical estimates to be compared to \(f_\lambda\). Nevertheless, we found that in the case of our experiments most of these corrections are insignificant and can be ignored. For clarity, we argue why this is the case in Appendix B. There we conclude that for chain-oscillations in water, and the orifices and flow-rates we investigated, the only significant correction is due to large amplitudes \(\varepsilon\).

In order to obtain an analytical solution for arbitrarily large amplitudes \(\varepsilon\), one would need to solve the full non-linear problem, for which there is no known solution to this day. However, Niels Bohr improved Rayleigh’s linear theory up to 2nd order in amplitude, meaning that only 3rd (and higher) order terms in amplitude are discarded. This yields an additional correction on the frequency of oscillation:

\[
\begin{align*}
    f_{\text{Rayleigh}} &= \frac{1}{2\pi} \sqrt{\frac{6\gamma}{\rho a^3}} \\
    f_{\text{Bohr}} &= \frac{1}{2\pi} \sqrt{\frac{6\gamma}{\rho a^3}} \sqrt{1 - \frac{37}{24} \left(\frac{b}{a}\right)^2} 
\end{align*}
\]  

(1.9)

Note that Bohr’s non-linear correction predicts the frequency to depend on the amplitude of the oscillation \(b\) (or the adimensional amplitude: \(\varepsilon \equiv \frac{b}{a}\)), which is a generic trait of a non-linear oscillation. Using the same measurements of the parameters \(a(Q)\) and \(b(Q)\) (as a function of the flow-rate), we can thus determine the frequency experimentally (1.6), as well as estimate it with the theoretical predictions (1.9). We can now compare \(f_{\text{Rayleigh}}\) to \(f_{\text{Bohr}}\), using the experimental measurement \(f_\lambda\) as a reference:
Using equations (1.6) and (1.9), we obtain the 3 different frequencies as a function of $Q$, as shown in Figure 1.9 for one particular orifice. The graphs corresponding to the remaining orifices can be found in Appendix B, as well as the profiles of the parameters of the chain-figure: $a(Q)$, $b(Q)$ and $\varepsilon(Q) = \frac{b(Q)}{a(Q)}$. As is plain to see in Figure 1.9, the frequency decreases with increasing flow-rate. This is due to the increase of $a$ and $\varepsilon$ with the flow-rate, which lowers the frequency as seen in equations (1.6) and (1.9). The good agreement of theory and experiment seems to confirm that chain-oscillations do indeed correspond to the expression of mode $n = 2$.

Additionally, it is clear that Bohr’s corrected formula agrees much better with the direct measurement, than Rayleigh’s. In fact, his approximation for this dependence seems to match the measured frequency within the given experimental uncertainties, even though it is only valid up to 2nd order in amplitude. This holds for the whole set of orifices we studied in our experiments, as can be confirmed in Appendix B. To show this in a more accessible way, in Figure 1.10 we plot a normalized residual sum of squares as a function of the orifice. To distinguish the orifices, we used their area, given that we did not use two orifices with the same area (see Figure B.3 in Appendix B).

Figure 1.9: Frequencies $f$ [Hz] as functions of the flow-rate $Q [\frac{1}{\text{min}}]$ in orifice 9. The experimental data corresponds to $f_\lambda$, which is used as a reference to compare $f_{\text{Rayleigh}}$ to $f_{\text{Bohr}}$.

Figure 1.10: Normalized RSS vs $A [\text{cm}^2]$ for all 13 orifices
The normalized residual sum of squares (RSS) is defined in the following manner:

\[
\text{normalized } \text{RSS}_{\text{Bohr}/\text{Rayleigh}} := \frac{1}{f_{\lambda_{\text{max}}}} \sum_{i=1}^{N} (f_{\lambda_{i}} - f_{\text{Bohr}/\text{Rayleigh}_{i}})^2 ,
\]  

(1.10)

where the sum over \(i\) denotes the sum of all data points (flow-rates) for a given orifice. That sum is then normalized with the square of the maximum value of the frequency, so that the summed discrepancies between the theory and experiment are adimensional and comparable to each other for different orifices. Would we not normalize these sums, then there would seem to be a trend in accuracy, since the absolute value of the frequency of chain-oscillations depends on the jet’s cross-sectional area (which in turn is proportional to the area of the orifice). The fluctuations of the RSS’s of the different orifices fluctuate because of statistical uncertainties in measurements, but above all because they comprise differently sized sets of data. In other words, the amount of data points for a given orifice, \(N\), is not constant and depends on the orifice (and lies between 18-29 data points).

Acknowledging this, the data indicates that Bohr’s corrections fit the experiment much better than Rayleigh’s, and about equally better for differently sized orifices. Put differently, Bohr’s model seems to approximate the phenomenological reality better, independently of the size of the orifice, for the conditions studied here. The only exception to this is orifice R10/7.5, for which the normalized RSS seems to indicate a better fit for Rayleigh’s isochronous (linear) estimate. Nonetheless, this was the orifice of our set with the least eccentricity (see Figure B.3), and hence the orifice which excites the smallest amplitudes of the liquid chain-oscillations. And for small amplitudes we expect the formulas to degenerate into each other, implying that the values of RSS_{Bohr} and RSS_{Rayleigh} should be the same for that particular orifice. Seeing as the discrepancy between them can be accounted for with the uncertainties in the measurements, this can be assumed to be the case with a large degree of confidence. In a nutshell, Figure 1.10 seems to imply that chain-oscillations are indeed non-linear oscillations, for which the frequency (period) of oscillation depends on the amplitude. In other words, chain-oscillations can only be regarded as isochronous oscillations for very small amplitudes. As such, this good agreement seems to corroborate Bohr’s non-linear corrections.

1.5.3 Discussion

In the case of a water jet surrounded by air, and for our experimental conditions, we find that the effects of gravity, the surrounding air and viscosity produce negligible corrections on the frequency of oscillation. In addition, although chain-oscillations are a 3D phenomenon, we find that the curvature in the radial plane dominates considerably over the curvature along the jet axis, which corresponds to the condition \(|ka| \ll 1\). This is indeed well satisfied in our experiments, except for a few data-points at low flow-rates in 3/13 of our orifices. Since this corresponds to a very small portion of our data-set, it was left to be discussed in Appendix B. There, we find that a correction in \(|ka|\) is needed to account for a small discrepancy in the measured frequency of those data-points.

Hence, our results suggest that chain-oscillations (in water) manifest close a laminar flow regime, and for small \(|ka|\). In this case, a good agreement is found between theory and experiment, when Niels Bohr’s non-linear corrections are taken into account. This suggests that, to a very good degree, chain-oscillations can be thought of as a non-linear oscillation of the 2D cross-section of a liquid jet, as the liquid flows along the jet axis. Moreover, chain-oscillations seem to indeed correspond to the normal mode \(n = 2\) of a capillary jet. Although Bohr’s estimate seems to match the measured frequency within the given experimental uncertainties, it is by no means exact. His estimate is valid up to 2\(^{nd}\) order in amplitude, so that it only contains the 1\(^{st}\) correcting term in amplitude, compared to Rayleigh’s linear analysis. Going up to arbitrary order in the theory should thus account for the small remaining discrepancy between Bohr’s estimate and the direct measurement. There are seemingly few confirmations of Bohr’s corrections. Still, a recent paper ([7] 2015) provided experimental evidence, that also seems to give Bohr credibility. The authors were able to verify the spectral component of \(\cos(4\phi)\) in a chain-oscillation at non-negligible amplitude. Furthermore, the measured ratio between the spectral components of the (small amplitude) \(\cos(2\phi)\) term and the additional \(\cos(4\phi)\) term matched the predictions of Bohr’s higher order corrections (within the experimental uncertainty). In our experiments we arrive at the conclusion, that the frequency seems to match the experimental data much better with Bohr’s higher order corrections. Thus, our results also seem to give Bohr credibility, but by corroborating his non-linear corrections on the frequency of chain-oscillations.

Regarding our experimental results, there is another issue that we want to discuss: as one can see from the frequency plot in Figure 1.9, the frequency as a function of flow-rate seems to be a quite complicated...
function. One might have expected a smooth relationship, however the frequency profile seems to be quite irregular. This is the case for all the used orifices (see Appendix B). The origin of those irregularities resides in $a(Q)$ and $b(Q)$, as they display un-smooth profiles as well. The simplest explanation would be that the experimental uncertainty in the measurement of those quantities was underestimated. However, repeating some measurements suggested that the uncertainties were well founded. This lead us to the conclusion that these irregularities are due to additional components of higher order modes. As one can observe in Figure 1.2, the ellipse is not the “ideal” cross-section of a chain-oscillation. Although an elliptical orifice can produce a liquid chain, the produced chain-figure will not correspond to a “pure” chain-oscillation. This means that instead of containing exclusively a $n = 2$ perturbation, the chain-figure we produced will also contain higher order modes. This is because there is a discrepancy between the ellipse and the “ideal” cross-section of chain-oscillations. Therefore, even though an elliptical orifice excites a large portion of mode $n = 2$, additional higher order components are also excited to account for that discrepancy. Given the symmetry of an ellipse, mode $n = 3$ should be suppressed and the first higher order term should correspond to a component of mode $n = 4$, as depicted in Figure 1.11.

\[ a(Q) \approx b(Q) \]

Figure 1.11: Illustration of the ellipse as a sum of different modes. Modes $n = 2$ and $n = 4$ are displayed at large amplitudes, to demonstrate that an elliptical orifice forcefully excites components of higher order modes. In Part (II) we show that these shapes correspond indeed to the respective modes at large amplitudes.

In fact, the presence of a component of mode $n = 4$ can be observed with the naked eye in Figure 1.12. This can be understood and confirmed in 2 distinct ways: either by looking at the prediction for the frequency of mode $n = 4$, or by considering the symmetry of an $n = 4$ perturbation around the jet — that is, its effect on the boundary shape $R(\phi, z)$.

\[ R(\phi, z) \]

Figure 1.12: Pictures of chain-oscillations at high amplitudes displaying a component of mode $n = 4$
Noting that mode \( n = 4 \) corresponds to a perturbation proportional to \( \cos (4\phi) \cos (kz) \), it produces deformations with local extrema both along the stretched and compressed axes of the chain-figure. These perturbations are seen in Figure 1.12 as 3 dimples on the chain-figure, which make the surface of the jet appear irregular. These would not be there if we had produced “pure” chain-oscillations, in which case the liquid chain would have a smooth surface. Note that the theory predicts that 
\[
\frac{f_n}{f_2} = \sqrt{\frac{60}{n^2}} = \sqrt{10} = 3, 1623... \approx 3.
\]
There should thus seem to fit about 3 periods of oscillation of mode \( n = 4 \) inside a period of a chain-oscillation (mode \( n = 2 \)). This is indeed the case, as seen in Figure 1.12, and explains why we observe 3 dimples.

Note that at small amplitudes these 2 oscillations never meet, since \( \sqrt{10} \) is irrational. Any orifice shape which does not correspond to the shape of a pure mode (like the ellipse) will carry extra components, which will oscillate in irrational proportions to the base mode. This means that the shape of the orifice is never again repeated in a cross-section of a chain-oscillation, unless it is a pure mode (although there are exceptions, e.g. \( f_3 = 2f_2 \)). Nonetheless, at large amplitudes the frequencies are displaced differently, because the factor of correction in amplitude depends on \( n \). Thus, at different flow-rates (amplitudes) the higher order modes intersect the chain-figure at different locations, contributing differently to the shape of the chain-figure. These higher order contributions deform the chain-figure, and are not being taken into account in the theory. The irregular profile of \( f(Q), a(Q) \) and \( b(Q) \) is thus a reflection of the \( n = 4 \) mode perturbing the chain-figure and our measurements. This again suggests that “pure” chain-oscillations correspond to the normal mode \( n = 2 \) of a capillary jet. To end with, note that we expect much purer vibrations, if the shape of a pure mode is used as the orifice.

1.6 Complementary experiment: “The formation of a jet”

The relationship between the orifice’s length-scale \( R_0 \) and the radius of the liquid chain \( a \) is still lacking in the present theory. In this section we will discuss why these differ and suggest how they can be related.

Our experiments have shown that the cross-section of the produced jet depends on the flow-rate \( Q \) and is considerably smaller than the cross-section of the orifice. This implied that the length-scale of the orifice differed from the length-scale of the jet, \( R_0 \neq a = a(Q) \). As a matter of fact, the parameters of the chain-oscillating jet were all found to depend strongly on the flow-rate. We believe this relates to a yet unresolved observation from Niels Bohr: Bohr first noted that the 1st wavelength of chain-oscillations seems to always be a bit shorter than what the theory predicts [2] and, as it seems, there is still no understanding as to what that means [3]. This has been attributed to the “presence of the orifice”, or in Bohr’s words, the “formation of the jet”.

Observing the shape at the exit of the orifice as in Figure 1.13, we see that the liquid does not exit at 90°. If it did, then this would be equivalent to perturbing a pendulum to some elongation and letting it go with no initial velocity, and the wavelength would be unperturbed. That is to say, that exiting at 90° would mean that all the orifice is doing is producing an initial cross-section of the jet, which has some elongation, but no initial radial velocity. But in reality, at the point where the liquid is expelled, the liquid jet makes an angle with the plane of the orifice, which depends on the flow-rate. We define this exit angle \( \theta_{\text{exit}} \) as the angle between the surface of the jet and the major axis of the elliptical orifice, as seen in Figure 1.13. This implies that the liquid shoots inward with some initial radial velocity, which explains the smaller 1st wavelength and period mentioned in previous references [2, 3]. This would then be equivalent to perturbing a pendulum to some elongation and letting it go with some initial velocity. Note that this also explains the diminished cross-section of the jet, compared to the orifice: for the cross-section of the jet to match the area of the orifice, the liquid should exit continuously at an exit angle \( \theta_{\text{exit}} = 90° \). However, since \( \theta_{\text{exit}} < 90° \), the jet’s cross-section is smaller than the orifices, implying that \( a(Q) < R_0 \). Thus, this also explains our earlier observation in this regard.
Figure 1.13: Chain-oscillations emanating out of an elliptical orifice at different flow-rates. Increasing flow-rate $Q$ and exit angle $\theta_{exit}$ from left to right. The exit angle $\theta_{exit}$ is defined as the angle between the major axis of the elliptical orifice and the surface of the jet, from a picture taken at $0^\circ$ using the convention of Figure 1.3.

As the liquid exits through the orifice, it goes from experiencing shear at the inside of the orifice and no surface tension, to feeling no shear and feeling surface tension on the free surface. Therefore, we expect the formation of a singularity due to an abrupt change in boundary conditions [5]. This singularity in the flow-profile is expected to manifest because the boundary condition itself is singular, as it changes abruptly over an arbitrarily small length-scale. In other words, the length-scale $L$ over which the boundary condition changes is so disproportionately small (compared to a typical length-scale of the system), that it can be assumed to vanish $L \to 0$. In this limit, there is an instantaneous change in the flow of the liquid at the moment it exits the orifice, which requires an infinite derivative at that point, and hence a singular behaviour. We thus expect the local behaviour of the flow-profile near the orifice exit to be universal, and its solution to be self-similar (scale invariant) as a function of the distance from the singularity. To make accurate theoretical predictions, one would need to solve for a singular flow profile around the orifice, taking into account the geometry and the conditions of the exterior and interior of the jet nozzle. In other words, one should be able to describe this singular behaviour through a local singular expansion of a potential flow solution around the orifice. Notwithstanding, as a 1st approximation, the liquid should follow the direction of the vector addition of these two forces:

$$\tan(\theta_{exit}) \simeq \frac{\sigma \eta}{\gamma \kappa} \bigg|_{r=D_{max}}.$$  \hspace{1cm} (1.11)

Here $\kappa$ represents the curvature of the orifice and $\sigma_\eta = \sigma_\eta(Q)$ the viscous shear stress with the orifice’s interior, which will depend on the flow-rate. Note that since our orifices are elliptical, the curvature of the orifice $\kappa = \kappa(\phi)$ varies from point to point. Therefore, we take a picture of the chain-figure at $0^\circ$ using the convention of Figure 1.3, and the curvature should thus be determined at the major axis, $r = D_{max}$. For a circular orifice of radius $R_0$ one would simply have $\kappa(\phi) = \frac{1}{R_0} = \text{constant}$. The prediction (1.11) seems to agree qualitatively with our observations: when there is no flow ($\sigma_\eta = 0$) the angle is $\theta_{exit} = 0^\circ$ and the water surface rests horizontally, and only in the limit $\sigma_\eta \to \infty$ does the exit angle approach $90^\circ$. In addition, seeing as $\sigma_\eta \propto Q$, we expect the exit angle to behave similar to $\theta_{exit} \propto \arctan(Q)$. As an example, in Figure 1.14 we display $\theta_{exit}(Q)$ for 3 of our orifices. The observed behaviour is indeed of the form of $\arctan(Q)$, suggesting that the 1st wavelength of chain-oscillations is always a bit shorter than the current theoretical prediction.
Figure 1.14: $\theta_{\text{exit}}(Q)$ in [°] as a function of $Q$ [L/min] for 3 distinct elliptical orifices. From light to dark green, these correspond to orifices 2, 6 and 9, respectively, with increasing cross-sectional area. The observed behaviour is of the form $\theta_{\text{exit}}(Q) \propto \arctan (Q)$, suggesting that $\theta_{\text{exit}} = 90^\circ$ is only reached in the limit $Q \to \infty$. 

As a matter of fact, this should hold for any geometry of the orifice, including circular orifices. This also indicates that the cross-section of a capillary jet is always smaller than the area of its orifice. In other words, the length-scale of the orifice is always larger than the length-scale of the jet. As (1.11) implies that $\theta_{\text{exit}}(Q) \leq 90^\circ$ and $a(Q) \leq R_0$, this corroborates our earlier observation of a smaller jet cross-section than the orifice’s area. Nonetheless, as of now we can only estimate the shear-rate, and therefore a thorough experimental investigation is still in need, as well as a more sophisticated theoretical approach. Using a singular flow solution, one should be able to determine $\theta_{\text{exit}}(Q)$, leading to the relationship between $R_0$ and $a(Q)$.

To end with, we claimed earlier that we expect much purer vibrations, if the shape of a pure mode is used as the orifice. However, the ideal shape would also have to take into account some corrections due to the initial velocity produced at the orifice due to the exit-angle $\theta_{\text{exit}}$, as observed in Figure 1.13. Since this angle depends on the flow-rate $Q$, an ideal orifice would only produce pure chain-oscillations at a given flow-rate. Still, we can expect much purer vibrations, if the shape of a pure mode is used as the orifice, instead of an elliptical orifice.

1.7 Additional experimental observations

In this section, we inspect the chain-figure more closely, underlining some of its finer traits. Later, we will be able to understand the observations made here better, when exploring the theory underlying the phenomenon.

At large elongations, chain-oscillations display round thick edges, with a quite thin liquid layer in between. As the flow rate is increased for a fixed orifice, the amplitude of the liquid chain-figure increases too, displaying these thick ends much clearer, as seen in Figure 1.15. At very large amplitudes, the oscillations seem to become asymmetric in time (or along the jet axis). As their amplitude is further increased, the chain-oscillations turn into a liquid sheet. At that point, the liquid spreads over such a large distance that the oscillations are not able to converge anymore. The liquid then spreads continuously, breaking up as a sheet. Interestingly, we found that the sheets form perpendicular to the orifice, meaning that half an oscillation still takes place. That is, after exiting the elliptical orifice, the liquid converges once into a circle (turning $90^\circ$), and only then spreading into a liquid sheet. This is a consequence of the exit angle $\theta_{\text{exit}}$ at the orifice. As argued previously, there is an initial in-shooting radial velocity at the orifice, which permits the liquid to converge once, before diverging and breaking up. In Video 1 one can observe how the chain figure evolves to a sheet over time, whilst the flow-rate is increased.
In *Figure 1.15* and 1.16, one can observe sheet behaviour: capillary waves, sheet flapping (due to the Squire instability), and rounder thicker edges (which break up at the rims due to the Rayleigh-Taylor instability), [6]. Notwithstanding, more research is needed in this regard and a good next step would be to measure how the drop-size distribution changes as chain-oscillations transform into liquid sheets.
Capillary waves are also excited close to the orifice, even at low flow-rates. Near the orifice, the expelled water converges inward, as chain-oscillations are set into motion. One can see that some capillary waves are excited in the process, which propagate on the water-jet’s surface. These are excited at the thicker edges of the in-shooting oscillation and propagate inward. Approaching the 1st nodal point of the chain-oscillation, these capillary waves seem to interfere at a certain angle, as seen in Figure 1.17.

Figure 1.17: Snapshots of chain-oscillations at high amplitudes near the orifice exit.

This very much resembles the interference pattern found in cusp tips. The best example hereof is the interference of light, which looks remarkably similar, as shown in Figure 1.18.

Figure 1.18: (a) Caustic lines (or cusps) and their formation in a circular vessel. (b) Cusp-tip interference patterns (also known as wave catastrophe).
To end with, we are keen to present some snapshots of the breakup of a chain-oscillating jet. If the oscillations damp out before the breakup-length of the jet is reached, the jet should be circularly symmetric at the pinch-off point. In that case, the jet is expected to break up due to the Rayleigh-Plateau instability, retaining a circular form. In our case, however, the oscillations experience low damping and can be seen to break up way before they reach the circular form. Owing to the Rayleigh-Plateau instability, the circularly symmetric perturbations grow superimposed with chain-oscillations all the same, as observed in Figure 1.19. In this case, where there is still some oscillatory motion at the point the jet breaks up, we believe that pinch-off is not symmetric (rotations in $\phi$).

Figure 1.19: Pictures of breakup in chain-oscillations at different angles: respectively $0^\circ$, $30^\circ$, $60^\circ$ and $90^\circ$.

Noting that the breakup of a circular jet is known to be up-down "asymmetric" around the breakup point, we would like to clarify what is meant here: thinking in cylindrical coordinates, the breakup of a circular jet is known to be asymmetric along the jet axis (in $\vec{e}_z$), but symmetric around it (in $\vec{e}_\phi$), [5]. In the case of chain-oscillations, pinch-off dynamics seem to be asymmetric in both $\vec{e}_z$ and $\vec{e}_\phi$. We believe that in the moment of pinch-off, the distorted chain-figure is not circularly symmetric (rotations in $\phi$), as observed in Figure 1.19. As a consequence, pinch-off events during chain-oscillations are not circularly symmetric (in general) and the remaining oscillatory motion is expected to propagate into the resulting droplets. Indeed, by magnifying some of the pictures of the breakup of the liquid chain, we could observe some of the produced droplets to be elongated:
Figure 1.20: Magnified snapshots of oscillating drops after breakup in chain-oscillations.

Observing the produced droplets in Figure 1.20, one again gets the impression that pinch-off is asymmetric in both $\vec{e}_z$ and $\vec{e}_\phi$, as described above. Perhaps more amusing is that this oscillatory motion seems to indeed propagate into some of the produced drops. It seems, the oscillatory motion is propagated up to the pinch-off point, which is unsurprising, since the oscillations were not damped before the singular pinch-off event. Although this was to be expected, a more thorough investigation is still in need; and naturally, more concrete scientific evidence.
Part II: On the normal modes of drops and jets

It was claimed earlier, that chain-oscillations are a normal mode of oscillation of a (capillary) liquid jet. And as such, we will need to dive deeper into the theoretical description of these modes, if we are to understand the more subtle aspects of the chain-figure. Nonetheless, this will be quite rewarding, as we will find many connections between the theory and our earlier observations. This second part of our investigation thus deals with the normal modes of capillary liquids, their connection to chain-oscillations, and some interesting surprises along the way.

2.1 Introduction

A theoretical description of the vibrations of a capillary jet was developed by Lord Rayleigh [1], and later improved by Niels Bohr [2]. A brief summary of their work is presented in Appendix A. Additionally, we have introduced the most important notions of these modes in Part (I) - 1.3 Underlying theory. This section should be taken as the extension of that introduction, in which we will be more rigorous in the mathematical description of these modes.

For \( a \) to represent the correct length-scale associated to the radius of the jet’s cross-section, the boundary shape \( R(\phi, z) \) should be defined such that its cross-section is \( A = \pi a^2 \). However, by integrating over the shape of a pure mode, \( R_n(\phi, z) = a + b_n \cos (n\phi) \cos (kz) \), we see that its cross-sectional area is:

\[
\begin{align*}
A_n &= \pi \left( a^2 + \frac{b_n^2}{4} \right) \quad \forall n \in \mathbb{N} \\
A_0 &= \pi \left( a^2 + \frac{b_0^2}{2} \right) \quad \text{for the case } n = 0 .
\end{align*}
\]

Since \( b_n \) is required to be small for the linear theory to be accurate, its contribution to the area can be ignored. However, to be more rigorous, the shape of a pure mode should be defined as:

\[
R_n(\phi, z) = a - \frac{b_n^2}{8a} + b_n \cos (n\phi) \cos (kz) \quad \forall n \in \mathbb{N} ,
\]

where in the case of \( n = 0 \) the constant term should be substituted by \( -\frac{b_0^2}{4a} \). In this way, the cross-sectional area is \( A = \pi a^2 + O(b_n^4) \approx \pi a^2 \). This can be seen to be the case, as \( a^2 \to \left(a - \frac{b_n^2}{8a}\right)^2 = a^2 - 2a \frac{b_n^2}{8a} + O(b_n^4) \).

The boundary shape \( R(\phi, z) \) in cylindrical coordinates, which describes the shape of a perturbed jet, is thus given by:

\[
R(\phi, z) = a + \sum_{n=1}^{\infty} \left( b_n \cos (n\phi) \cos (kz) - \frac{b_n^2}{8a} \right) + b_0 \cos (kz) - \frac{b_0^2}{4a} \quad \forall n \in \mathbb{N} ,
\]

Again, this is only valid up to small deformations, as it is the solution of the linear theory, and it only takes the effect of surface tension \( \gamma \) into account. Since the constant terms only bring about quantitative changes and not qualitative, this still represents a normal mode expansion for the perturbations of a cylindrical jet.

Chain-oscillations can now be defined to correspond to the normal mode \( n = 2 \). And since \( \omega_1 = 0 \), they represent the fundamental (or 1st) mode of vibration of a liquid jet (or 2D drop). This aids in understanding why they are observed so often in nature: they are the easiest vibrational mode to excite. An ideal chain-oscillation should thus be understood as the expression of a pure \( n = 2 \) mode on the capillary jet. However, any vibration with a large \( n = 2 \) component (and some additional higher order perturbations) will display a very similar behaviour, which for most purposes will be indistinguishable from a ”pure” chain-oscillation. Therefore, any shape that excites a large portion of this mode can in practice be seen as a chain-oscillation. In this regard, we note that \( n = 2 \) is the only normal mode capable of stretching the circular cross-section (or 2D drop) into an elongated oval shape (see Figure 1.5). This implies that any stretched initial cross-section of a jet, will induce these oscillations, as it will forcefully excite a large portion of this mode. This explains, for instance, why these oscillations are excited in pouring a liquid:
Figure 2.1: Pouring water from a bottle. Chain-oscillations are excited because the initial cross-section is stretched (highlighted dashed lines).

As a form of visualizing the oscillation cycle of a 2D drop, (or cross-section of 3D chain-oscillations), one can think of the points of maximum curvature (positive and negative) to push inward and outward respectively. Oscillating through the equilibrium shape, the circle, it thus seems that the shape rotates 90°, as seen in Figure 1.2. In fact, as argued previously, this will be true for each mode, which will appear to “rotate” $\frac{180°}{n}$ after each half oscillation, as the sinusoid $\cos(kz)$ changes sign. There are excellent videos demonstrating these dynamics for quasi-2D drops: (drop oscillating on a plate in mode 2, [14]; sound-levitated drop oscillating, [10]). With quasi-2D drops we mean 3D drops which behave effectively as 2D drops. We have claimed that at long wavelengths the cross-section of the normal modes of a cylindrical jet degenerate into 2D drops. For 3D drops this is not exactly the case, however, the modes of a 2D drop can still be seen to qualitatively be contained as the cross-section of the modes of a 3D drop. An example hereof are the “star-shaped” Leidenfrost drops presented in Figure 2.2.

Figure 2.2: Normal modes of Leidenfrost drops [14]. These can serve as an example of 3D drops behaving as quasi-2D drops. (a) - (f) respectively correspond to expressions of modes $n = 2$ to $n = 7$. 
2.2 3D vs 2D modes

The normal modes of a drop (as described in 3D spherical coordinates) show a remarkable geometric similarity to the normal modes of a cylindrical jet. More concretely, if we examine a jet and a drop which are deformed into the same pure mode $n$, their cross-sections will look alike. Hence, one could (almost) think of the normal modes of a jet as a series of drops oscillating in the same mode, as they move along the jet axis. A simple way to understand why that is, is to examine the simpler problem of the normal modes of a 2D drop in polar coordinates. As shown in Appendix C (in section C.6), the normal modes of a 3D drop and jet can be thought of as generalizations of the normal modes of a 2D drop. These generalizations have mild quantitative differences, but are qualitatively the same. In fact, one can understand the 3D modes to degenerate into the 2D modes for specific conditions. In the cylindrical case, we find that the jet’s cross-section behaves exactly like a 2D drop in the limit $|k_0 a| \to 0$, and if the jetting speed $v_z$ is uniform (laminar flow). As mentioned earlier, these correspond to the conditions of our experiments, which further motivates the study of these modes in 2D.

2.3 Normal modes of a 2D drop

In this section, we are concerned with the description of the normal modes of a 2D drop in polar coordinates. As argued in the last section, this presents a much simpler way to understand their generalizations in 3D. With the following description, we arrive at the same linear (small amplitude) solution as the previous references, but using yet a different approach. A more detailed derivation can be found in Appendix A. Instead of the Lagrangian method used by Rayleigh [1] and the time-independent solution derived by Bohr [2], our derivation is based on a time-dependent Stokes Flow. In using this approach, we find some promising new results that build on their work, and we are able to give an additional interpretation to the phenomenon. These new considerations will be the subject of the next few sections.

We assume that the Reynolds number is small, which allows us to ignore the convective derivative, yielding a time-dependent Stokes-Flow. Adding the incompressibility condition, the pressure profile is seen to be harmonic, as it is forced to obey the Laplace-Equation (or harmonic equation):

$$\begin{cases}
\rho \frac{\partial \vec{u}}{\partial t} = -\vec{\nabla} p + \eta \nabla^2 \vec{u} \\
(\vec{\nabla} \cdot \vec{u}) = 0
\end{cases} \implies \nabla^2 p(\vec{r}, t) = 0 \quad . (2.4)
$$

Note that the partial time derivative is kept, as there is an additional source of temporal dependence: surface tension, at the boundary interface between the water and the surrounding air introduces both a restoring force (for shapes deviating from the cylinder) and an according pressure profile in the bulk of the fluid phases; mostly inside the liquid, the inner phase, but also in the surrounding air, or outer phase, as Bohr showed [2].

Note further, that when the Reynolds number cannot be ignored, equation (2.4) generalizes to:

$$\nabla^2 \left( p + \frac{1}{2} \rho \vec{u}^2 \right) = 0 \quad (\text{as } \vec{f} = (\vec{\nabla} \cdot \vec{f}) \quad ) \quad , \quad (2.5)
$$

where $\vec{f}$ is an additional bulk-force that can be applied on the fluid. Notice that this imposes a Poisson-Equation on the pressure, as a generalization of the Laplace-Equation. We can solve the harmonic equation by the method of variable separation, obtaining a general expression for the pressure:

$$\nabla^2 p = 0 \implies p(r, \phi) = \left( \alpha \phi + \beta \right) \log(r) + \sum_{n \in \mathbb{Z}} r^n \left[ A_n \cos (n \phi) + B_n \sin (n \phi) \right] \quad . \quad (2.6)
$$

Here, $A_n$ and $B_n$ are constants which are not to be confused with the cross-sectional area $A$. Note that it is known, that if a solution to the Laplace-Equation (more generally, the Poisson-Equation) can satisfy all possible boundary conditions, then it is unique and the full general solution. Thus, we have a normal-mode expansion for the pressure of any non-turbulent, incompressible fluid! Let us now look at these modes individually:

- The $\log(r)$ term is known to describe sinks/sources ($\beta \neq 0$) and vortices ($\alpha \neq 0$), or combinations of both.
- $n = 0$ describes constant pressure and is therefore the equilibrium shape, the circle.
• \( n = 1 \) is the only term that can describe movement about the center of mass, since it is the only mode able to satisfy \( \nabla p = \text{constant} \neq 0 \) everywhere.

• \( n \geq 2 \) describe symmetric deformations, that can produce "star-shaped" oscillations and polygonal patterns.

• \( n \leq -1 \) describe the outer phase of their positive counterpart. They become important for hollow polygonal patterns (or "star-shapes").

Let it be noted that all terms in the sum over \( n \), if present, produce movement in both the radial and angular directions. Additionally, there are 2 ways to have either only radial flow, or only tangent flow (in the direction of \( \phi \)), which correspond to the logarithmic terms. Combinations of these modes are also allowed. Now we want to know how the surface of a drop deforms, if the pressure inside it corresponds to a particular mode. Therefore, we will have to impose that the pressure drop at any point on the surface matches with the pressure of a mode \( n \). Due to surface tension, the Young-Laplace pressure drop determines the equilibrium shape of a 2D-drop to be a circle. Since the equilibrium shape already has a constant pressure drop across its surface, we have to add that constant to each term. This way, the pressure due to a mode corresponds to the difference in (extrinsic) curvature, \( \kappa \), from the equilibrium shape (the circle):

\[
\begin{align*}
\gamma \kappa \big|_{r=R(\phi)} &= C + A_n r^n \cos (n \phi) \big|_{r=R(\phi)}, \\
C &= p_{\text{initial}} = \gamma \kappa_0 = \frac{\gamma}{a}.
\end{align*}
\]  

(2.7)

Note that from the \( \sin (n \phi) \) and \( \cos (n \phi) \) terms we may ignore one and keep the other, without loss of generality, as they correspond to rotations of the same mode.

\[
\begin{align*}
\gamma \kappa[R(\phi)] &= \frac{\gamma}{a} + A_n R^n(\phi) \cos (n \phi) \\
\Rightarrow A_n R^n(\phi) \cos (n \phi) &= \gamma [\kappa[R(\phi)] - \kappa_0].
\end{align*}
\]

(2.8)

Introducing the curvature operator for a parameterized curve in polar coordinates, we end up with:

\[
\frac{A_n}{\gamma} R^n(\phi) \cos (n \phi) = \frac{R^2(\phi) + 2R^2(\phi) - R(\phi)\dot{R}(\phi)}{[R^2(\phi) + \dot{R}^2(\phi)]^{\frac{3}{2}}} - \frac{1}{a},
\]

(2.9)

where the dot represents a derivative with respect to \( \phi \). We can now try solving this equation for \( R(\phi) \), using the Ansatz:

\[
R(\phi) = a + \delta R(\phi) \equiv a [1 + \xi(\phi)].
\]

(2.10)

The non-linear differential equation (2.9) was solved numerically in Mathematica, producing the following shapes:
The \( n = 1 \) shapes, which depict movement of the drop about its center of mass, have (at least) 3 solutions. The first corresponds to the surface not deforming at all. This is possible when all particles of the drop feel a force in the same direction, as in the case of a spherical droplet feeling a gravitational force (but no air resistance). The other shapes correspond to 2 different ways, for the boundary to deform in such a way, that the curvature induces only a force about the center of mass (in the drawn cases in the direction of the \( y \)-axis). The kinks in those 2 shapes should really be smooth extrema, however, we will leave them be for now, as we shall discuss these shapes further on.

For \( n \geq 3 \) we get polygonal shapes rounded off at the vertices, which correspond to the “star-shapes” of drops, as is referred to them in the literature [9, 10, 11, 12, 13, 14, 15, 16].

By symmetry, we can see that mode \( n = 2 \) is our desired shape of chain-oscillations, at least in 2D. It is undoubtedly a normal mode of oscillation of a liquid cylinder (or jet). Since \( n = 1 \) corresponds to the movement about the center of mass, only \( n \geq 2 \) can describe symmetric perturbations of the circular shape, which are the only modes capable of producing oscillations. Therefore, chain-oscillations \((n = 2)\) represent the fundamental or \( 1^{st} \) mode of vibration.

Continuing, we proceed as in Appendix A, ignoring viscosity and keeping only linear terms, which yields the general solution of a 2D oscillation of a pure mode \( n \), at small amplitudes:

\[
\begin{align*}
R(\phi, t) &= a - \frac{b^2}{8a} + b \cos (\omega_n t) \cos (n\phi) \\
u_r(r, \phi, t) &= -b \omega_n \left( \frac{r}{a} \right)^{n-1} \sin (\omega_n t) \cos (n\phi) \\
u_\phi(r, \phi, t) &= b \omega_n \left( \frac{r}{a} \right)^{n-1} \sin (\omega_n t) \sin (n\phi) \\
\rho(r, \phi, t) &= \frac{\gamma}{a} + \left( \frac{n^2 - 1}{n^2} \right) \frac{b}{a} \left( \frac{r}{a} \right)^n \cos (\omega_n t) \cos (n\phi), \\
A_n &= \frac{(n^2 - 1)}{a} \frac{b}{a^{n+1}} \\
\omega_n &= 2\pi f_n = \sqrt{\frac{n(n^2 - 1)\gamma}{\rho a^3}},
\end{align*}
\]
where in this laminar regime $\omega_n t = k z$, so that $v_z = \frac{z}{t} = \frac{\omega_n}{k}$. The term $-\frac{b^2}{\alpha}$ is introduced in $R(\phi, t)$ to ensure that to leading order, the area of the cross-section is $\pi \alpha^2$. This solution represents an undamped harmonic oscillator, displaying sinusoidal oscillations with constant amplitude and frequency. The perimeter of the 2D drop oscillates, while its area remains unaltered. Note that the velocity profile oscillates in quadrature to the pressure profile and the boundary shape. Using this approach, we find that both the pressure and the flow profile are scale invariant power laws in the radial coordinate. As such, the motion and shape of these normal modes is expected to be self-similar in this linear regime. Combining results with both previous references, the same shape of chain-oscillations can be obtained for:

- 3D, cylindrical coordinates
  - standing waves on a liquid cylinder [1]
  - stable stationary configuration of the surface of a liquid cylinder [2]
  - dynamical oscillations on a liquid jet at steady initial conditions (this work)
- 2D, polar coordinates (if $|ka| \ll 1$)
  - a slice of liquid (2D drop) oscillating as it flows uniformly in the direction of the cylindrical axis ([2] and this work)

Thus, chain-oscillations can be understood as the natural shape to produce any of the above situations, for a perturbation that is “purely” $n = 2$. In fact, this holds for every mode $n \geq 1$, and for any linear combination thereof, at small enough amplitudes (in this linear regime).

### 2.4 Additional theoretical results

In this section, we explore the theory a bit further, stepping away from the simplifications taken previously. As such, we will discuss and re-interpret the normal modes in question, and build on the previous investigations on this subject.

#### 2.4.1 The Geometry of the Modes

As we have argued before, the shapes of the 2D polar modes are very close to the 2D cross-sections of their generalizations in 3D at small amplitudes (spherical and cylindrical coordinates). Notwithstanding, even when they become different shapes, they are qualitatively (and dynamically) still the same. In addition to that, their unique geometry will be a powerful tool, due to symmetry arguments, and because they are normal modes. This is why this section will focus on building some intuition for 2-dimensional harmonic modes. All the modes are described quite accurately by the boundary shape:

$$R(\phi) = a + b \cos (n \phi + \phi_0) = a \left[ 1 + \varepsilon \cos (n \phi + \phi_0) \right], \quad \varepsilon := \frac{b}{a} .$$

(2.12)

Here $\phi_0$ is a constant phase that determines the orientation of the shape. We also introduce an adimensional amplitude $\varepsilon$. In fact, this shape is the exact analytical solution in the case of an infinitely small amplitude, $\varepsilon \approx 0$. Note that for simplicity we will leave out the constant term $-\frac{b^2}{\alpha}$ in this section. This can be done without loss of generality, as this term does only changes the results quantitatively, but not qualitatively. From the geometries and symmetries alone, it is not hard to see, that the following statements are true:

- The 2 deformed shapes of mode $n = 1$ are the only ones capable of inducing/resisting a movement about the center of mass of the drop.
- The shape of mode $n = 2$ (chain-oscillation) is the only one capable of stretching the drop (circle) around its center of mass.
- The shapes of modes $n \geq 2$ are the only ones capable of producing oscillations, where each successive mode will oscillate at a higher frequency.

Since we are matching the harmonic pressure mode at the boundary with the pressure excess produced by a curved surface at the boundary, we are also determining the normal modes of surface tension. And because surface tension carries the fortuitous physical interpretation of energy per surface area, as a very fortunate side-bonus, we can also claim that the shape of these modes corresponds to the most compact shapes with a particular symmetry and purpose/effect:
Having non-zero components of the modes $n \geq 1$ will always increase the surface area of the drop (perimeter in 2D), with respect to its equilibrium, the sphere (circle).

The modes $n = 1$ define the most compact shapes which produce a net force about the center of mass. Because they are the most compact shapes, this net force is the only effect that they produce on the drop.

The mode $n = 2$ defines the most compact shape, which stretches a drop/sphere around its center of mass. Therefore, the stretching effect is the only effect that it produces.

In general, a pure mode $n$ defines the shape which deforms a sphere (circle) symmetrically around its center of mass $n$ times, with the least amount of additional surface area (perimeter). Thus, these modes represent the class of most compact geometries with $n$ bilateral symmetries in 2D.

This goes hand in hand with Rayleigh’s calculations. Compactness is a quite complicated mathematical optimization problem and we happened to stumble upon the general solution (or at least the recipe for it in 3D). Due to this, one should already be expecting many examples of these exact shapes appearing in physics and biology in general. In fact, one would usually (in 2D) define the compactness of a shape to be: $\mathcal{C} = \frac{A}{\mathcal{P}}^2$. Here, $A$ is the Area and $\mathcal{P}$ denotes the perimeter surrounding $A$. Note that $A$ is constant and that $\mathcal{P}$ should oscillate at the rate $2f_n$. As one can see, the vibrations of these modes can also be thought of as being oscillations in compactness. Of course, the equilibrium shape (circle in 2D) corresponds to the points of minimum compactness, $\mathcal{C}$.

Let us now take a better look at the shape of these modes and compare them at different elongations (or phases). In doing so, we show that there are specific elongations (or amplitudes) of interest, one where the modes resemble polygons, and another where they look like “star-shapes”. These can be understood (or even defined) to correspond to conditions imposed on the curvature operator. Using (2.8) and since

$$\cos (n\phi) \bigg|_{\phi = \frac{\pi}{n}} = \cos (\pi) = -1 \ ,$$

we have:

$$\gamma \kappa [R(\phi)] = \frac{\gamma}{a} - A_n R_n(\phi) \bigg|_{\phi = \frac{\pi}{n}} = \varepsilon \ .$$

We will thus introduce the *polygon*-phase and the *star*-phase in the following way:

$$\begin{align*}
\kappa (\phi = \frac{\pi}{n}) = 0 & \implies A_n a^n \simeq \frac{\gamma}{a} \implies \varepsilon_{\text{polygon}} = \frac{1}{(n^2 - 1)} \\
\kappa (\phi = \frac{\pi}{n}) \leq -\frac{\gamma}{a} & \implies A_n a^n \geq \frac{2\gamma}{a} \implies \varepsilon_{\text{star}} \geq \frac{2}{(n^2 - 1)} ,
\end{align*}$$

(2.14)

where we used $R (\phi = \frac{\pi}{n}) \simeq a$ and $A_n \sim \frac{(n^2 - 1)\gamma}{a}$ from the linear solution of (2.11). In other words, at those precise amplitudes, the shape of the mode will resemble a polygon as best as possible (or a star-shape, respectively). This can be appreciated in the following figure:
Note that the conditions in (2.14) are only approximate conditions, and so are the shapes of those phases we depict here. If we take the energy of the deformation of each mode to its maximum amplitude (∞ phase), then this will produce kinks, that is, points of infinite curvature. Although there is no analytical solution for these shapes yet, they seem to be related to Epicycloids, as the ∞-phase of mode $n = 1$ resembles a caustic line (or cardoid). It would be interesting to simulate what happens at such extreme amplitudes, that the drop breaks. Perhaps under certain conditions it would be possible to excite these oscillatory modes up to an amplitude where the drop would break up. In an ideal case, one should expect $n$ equally sized drops to be produced (most likely along with some smaller satellite droplets).

### 2.4.2 Modes in the non-linear regime

There seem to be solutions to the non-linear differential equations describing the shapes at higher amplitudes. In other words, generalizations of the small amplitude solutions. We can use the same Ansatz as before,

$$R(\phi) = a + \delta R(\phi) = a[1 + \xi(\phi)] ,$$  \hspace{1cm} (2.15)

where $\xi(\phi)$ is now a complicated function to describe analytically. At inflection points it should satisfy:

$$\kappa(\phi_{inf}) = 0 = R^2(\phi) + 2\dot{R}^2(\phi) - R(\phi)\ddot{R}(\phi) ,$$  \hspace{1cm} (2.16)

and at extremal points:

$$\dot{R}(\phi_{extr}) = 0 \implies \frac{A_n}{\gamma}R^n(\phi) \simeq \frac{R^2(\phi) - R(\phi)\ddot{R}(\phi)}{\dot{R}^2(\phi)} - \frac{1}{\dot{a}} ,$$  \hspace{1cm} (2.17)

implying:

$$\begin{cases} \ddot{R}_n(\phi) \simeq \left[1 - \frac{A_n}{\gamma}R^{n+1}(\phi) - \frac{R_n(\phi)}{a}\right]R_n(\phi) \\ \ddot{\xi}_n(\phi) \simeq -a\left[\xi_n(\phi) + \frac{A_n\alpha^{n+1}}{\gamma}(1 + \xi_n(\phi))^{n+1}\right](1 + \xi_n(\phi)) \end{cases} .$$  \hspace{1cm} (2.18)

This looks similar to the differential equations satisfied by some elliptical functions. In other words, the differential equations satisfied by elliptic functions are generally of the form:
\[ \begin{align*}
\dot{y}(\beta) &= \text{polynomial}(\beta) \\
(\dot{y}(\beta))^2 &= \text{polynomial}(\beta).
\end{align*} \]  

(2.19)

In fact, in 2019 it was shown how near the extrema, the radial function approximates the Jacobi elliptic function \( dn(\xi, \kappa) \) in rotating hollow polygonal patterns in liquid nitrogen [22]. This will be illustrated in Part (III) of this document. In that case the equations satisfied by the radial function are not precisely the same, but very similar. The oscillations in time are likewise only isochronous as a 1st approximation (small amplitudes). Using Navier Stokes, one can argue that the oscillations in time should also resemble elliptic functions. Since \( p_n \propto r^n = a_n^{1+\xi_n(\phi, t)} \), we can say, using the time dependent Stokes-Flow:

\[ \rho \partial_t \vec{u} \simeq -\nabla p \implies \partial^2_t \xi_n(t) \propto [1 + \xi_n(t)]^{n-1}. \]  

(2.20)

The reader is invited to note here, that elliptic functions are still periodic, but unlike \( \sin x \) and \( \cos x \) they do not need to be symmetric anymore and can assume more intricate forms of oscillation. However, a more complicated, generalized, function \( \xi(\phi) \) should still have a periodicity of \( \frac{2\pi}{n} \). This can be seen in the equation describing the perturbation of the curvature (2.8), which states that \( \kappa(\phi) \propto \cos(n\phi) \). Fourier-expanding, this would imply that:

\[ \xi_n(\phi) = \sum_{i=0}^{+\infty} \alpha_i(n) \cos(in\phi + \phi_0). \]  

(2.21)

Note that this is consistent with Niels Bohr’s results at finite amplitudes, which our experiments seem to corroborate. Now the question arises, whether these modes stop becoming normal to each other, or if they “rotate” into each other in a way that maintains their orthogonality. In any case, since each generalized mode can only have an integer multiple of its angular dependance (\( \sum_{i=0}^{+\infty} \alpha_i \cos(i\phi) \)), a pure \( n = 2 \) mode, for instance, will never have any component in \( \cos(3\phi) \). This also means that we can always distinguish mode \( n \), since its 1st (and largest) component will be \( \cos(n\phi) \). This is at the heart of why the modes still behave qualitatively the same at large amplitudes. If we represented the Fourier-expansion of each mode, as a matrix holding the components of each sinusoidal term for each mode, this would yield a triangular matrix. This means that indeed, the modes can still be a valuable tool, even after degenerating into each other: because they are still linearly independent, an orthogonal (orthonormal) basis can always be constructed.

Another quite interesting question is whether the potential energy of mode \( n = 1 \) and its frequency are still damped at large amplitudes, seeing as they are damped for small amplitudes. The fact that we find deformed shapes at higher amplitudes for \( n = 1 \), suggests that there is indeed an excess of surface area, and thus some potential energy. However, at this point there is not much we can say about the frequency of these modes, and whether it is still damped. That is, should \( f_1 = 0 \) for larger amplitudes? In spherical coordinates, this is equivalent to asking: does a drop wiggle? Last but not least, to better understand the dynamics of these modes near convection (or for increasingly larger Reynolds numbers), we can try solving for \( \nabla^2(p + \frac{1}{2}\rho v^2) = 0 \) by calculating \( \nabla^2(\frac{1}{2}\rho v^2) \) for the solution of \( \nabla^2 p = 0 \) and adding it as a perturbation. This might present us with an insight on stability, and given the geometry of our system, in which regions or points instabilities and potential singularities might develop. In any case, it is curious to think about what

\[ \nabla^2(p + \frac{1}{2}\rho v^2) = \nabla^2(p + e_{\text{kin}}) = 0 \quad \text{really means:} \]

If \( \nabla^2 f = 0 \) is true in a certain region of spacetime, that means that \( f \) has no local extrema (one can think about it as ‘sources’ of the function \( f \)). \( f \) is then a harmonic function, whose maximum and minimum values lie at the boundary of that same region. Thus, \( f \) is just interpolating between the Boundary conditions and the conditions in the bulk. This is why the Laplacian describes static equilibrium conditions so naturally. In an abstract sense, this seems to be similar to a holographic principle: knowing the theory (\( \nabla^2 f = 0 \)) and the state of the boundary, one knows the state of all points in the bulk, since the theory determines them. Put simply, the condition \( \nabla^2 p = 0 \) just ensures that all points inside the liquid are always exactly in phase with the boundary at any moment in time. They always feel the pressure that the boundary shape would impose on them in an equilibrium situation, but instantaneously at any time. Naturally, this is only true, if the speed of sound inside the liquid is fast enough to travel to every point and regulate the pressure induced by a change in the boundary. As a consequence thereof is that there should be no convection! And thus, \( \frac{e_{\text{kin}}}{p} \ll 1 \), which, again, means that the kinetic energy can “dephase”, or “disrupt” these oscillations because of the increasing convection.
2.4.3 The modes $n = 1$

One of the most striking results of this work is the fact that it was noticed that there are (at least) 2 shapes of mode $n = 1$ that do deform the drop’s surface, whereas before this mode was assumed to displace the jet (or drop) as a whole, “without alteration in the form of the boundary” [1]. Hence, we will now revisit the modes corresponding to $n = 1$, reason why and when these solutions make sense, and argue that (generalizations of) these shapes should also be found in the 3D case (cylindrical and spherical coordinates).

In polar coordinates, the static equation which determines the boundary shape of a mode (the Young-Laplace equation), reads:

$$
\begin{align*}
A_1 R(\phi) \cos \phi &= \gamma (\kappa |R(\phi)| - \kappa_0) \\
A_1 R(\phi) \cos \phi &= \frac{R^2(\phi) + 2R^2(\phi) - R(\phi) \dot{R}(\phi)}{[R^2(\phi) + \dot{R}^2(\phi)]^2} - \frac{1}{a}
\end{align*}
$$

(2.22)

Three distinct shapes were found as numerical solutions to this differential equation, using Mathematica. The first corresponds to the movement of the 2D drop as a whole, as suggested by Rayleigh. And the additional two (new) shapes, as defined in Figure 2.5, will henceforth be designated as the “cusp-like” shape, and the “drop-like” shape:

![Figure 2.5: The “cusp-like” shape on the left, and the “drop-like” shape on the right.](image)

Note that by Taylor-expanding $R(\phi)$ in $\phi$ it can be shown, that for arbitrary $n$, $R(\phi)$ should have a local extremum at the points where $\cos(n\phi)$ (or $\sin(n\phi)$) have local extrema. Thus the shapes found here with Mathematica should not kink, but form a smooth extremum. This can be followed to the fact that Mathematica is using a polygonal numerical approximation, which is doomed to blow up at the domain end. More efforts to solve this with a Fourier analysis will be made in the future. Notwithstanding, the extrema for the shapes in Figure 2.5 are expected to develop over a very small length-scale, meaning that they should look very similar to kinks. In addition cusps seem to be able to form in surface tension dynamics at these extrema (for instance in shocks), due to non-linear dynamics. This is well illustrated by the work [18]. Let it also be said, that although the caustic line does not appear to be a solution to the differential equation, its tip, the cusp is known to be a natural shape for viscous interfaces. And given that viscosity becomes important at small scales, a capillary drop could very well “cusp” naturally. In any case, $R_1(\phi) = a [1 + \varepsilon \cos(\phi)]$ still shows the desired behavior for big enough $\varepsilon \leq 1$. 

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Figure 2.6: $R_1(\phi) = a \left[ 1 + \varepsilon \cos (\phi) \right]$ respectively for $\varepsilon = 0, \ 0.25, \ 0.5, \ 0.75, \ 0.95$

Since $R_n(\phi) = a \left( 1 + \varepsilon \cos n\phi \right)$ seems to qualitatively still reproduce the shape of all the modes with $n \geq 2$ for large amplitudes, this can be expected to hold for $n = 1$ as well. Alternatively, we could generalize $R_1(\phi)$ to a Fourier-sum, and solve the differential equation perturbatively for higher amplitudes, giving:

$$R_1(\phi) = a \left( 1 + \varepsilon \cos \phi + \alpha_2 \varepsilon^2 \cos 2\phi + \alpha_3 \varepsilon^3 \cos 3\phi + \ldots \right).$$ \hspace{1cm} (2.23)

As the exact form of the modes for large amplitudes is still unknown, here are 3 (random) examples of how the shape of $(1 + \varepsilon \cos \phi)$ could change with higher order terms:

Figure 2.7: $R(\phi) = \left\{ 2 + \sin \phi + \frac{\cos 2\phi}{2}; \ 2 + \frac{\sin \phi}{2} + \frac{\cos 2\phi}{2} - \frac{\sin 3\phi}{5}; \ 3 + \frac{4\sin \phi}{5} + \frac{6\cos 2\phi}{5} - \frac{2\sin 3\phi}{5} \right\}$ respectively from left to right

As one can see, qualitatively the shape does not change, again suggesting that there can be deformed solutions for $n = 1$ at large amplitudes. Let us come back to the 2 shapes in question. Intuitively, both seem to produce a net force in a particular direction. Nonetheless, there is a subtlety in how these shapes solve the differential equation (2.22): as can be seen in Figure 2.6, the deformed shapes not only deform, but also shift their center of mass about the origin. This relates to the curvature operator in polar coordinates, which is sensitive to the coordinate $R(\phi)$. This is also why shifting the circle about its center of mass satisfies the differential equation at small amplitudes. However, looking at the differential equation from the point of view of the curvature, $\kappa(\phi) \propto \cos (\phi)$. Thus, if we imagine calculating the curvature in the center of mass of the deformed shape, then the drop should deform into a sort of heart-shape, which would correspond to a combination of both the deformed shapes together. Therefore, we surmise that there can be more solutions to the deformed shapes of mode $n = 1$, corresponding to combinations of the "cusp-like"- and "drop-like" shapes. In fact, "heart-shaped" drops can be observed at pinch-off events of capillary jets.
As can be seen in the bottom right corner of Figure 2.8, pinch-off dynamics in capillary jets can excite these modes. This is not that surprising, as the drop needs to feel a force about its center of mass to detach from the neck. Arguably, it needs to excite mode \( n = 1 \). Now that we have discussed the details surrounding the deformed shapes of Figure 2.5 that were found numerically, we will proceed to argue why they should be expected to exist and when.

All of the 3 shapes in Figure 2.9 solve the differential equation defining the mode \( n = 1 \), which characterizes movement about the center of mass. The most left shape, which corresponds to the movement of the whole shape without deformation thereof, was found by Rayleigh [1], and thought to be the only possible solution. But in equation (2.22), we are essentially asking how the surface of the drop has to deform, to match with the pressure induced by a mode at the boundary; just as in all higher order modes, see (2.8). Therefore, there should be solutions of how a drop can deform to accommodate, produce, or withstand a constant force about its center of mass. Nonetheless, in polar coordinates, shifting the circle about the origin, will (for small amplitudes) satisfy the differential equation. This is equivalent to saying \( R_1(\phi) = a \left( 1 + \varepsilon \cos(\phi) \right) \), which for small \( \varepsilon \) will shift the circle about the origin (see Figure 2.6, for \( \varepsilon = 0.25 \)). A question arises here, namely if this should be allowed (since the original goal was to determine how the shape deforms to produce a certain effect), or if in this particular case the curvature needs to be calculated in the center of mass of the shape. The answer, a quite standard outside of the box answer, is: all 3 shapes are valid, depending on the boundary
conditions!

As an example, gravity clearly exerts a force on a liquid drop falling from the sky. Neglecting air resistance, the drop would not deform, as gravity acts simultaneously on all particles that compose the fluid, including the boundary. Introducing air resistance, the (main) effect of the flow around the falling drop will be to produce a constant resisting force in the precise opposite direction as gravity. However, due to the nature of the resulting flow profile, this will not act the same way at all points of the drop’s boundary. Consequently, the drop will deform into the "cusp-like" shape from below, as there it builds up much more pressure due to the air flow, than the pressure that is released from the upper part of the drop. So, one could understand the "cusp-like" shape as the natural deformation of pushing on a drop from one side. And similarly, the "drop-like" shape corresponds to pulling a drop on the opposite side, while a "heart-like" shape corresponds to an equal combination of both effects simultaneously.

Hopefully, with this example it becomes clear, that all 3 shapes and the "heart-like" shape can manifest, depending on the conditions at the boundary. Therefore, we believe that the movement about the center of mass of a drop can assume any combination of the 3 shapes displayed above in Figure 2.9. Although we did not prove that these are the only solutions of the differential equation (2.22), it seems straightforward that if we allow for the drop to shift about the origin, there can be solutions of the type of the "cusp-like" and "drop-like" shape. And if we fix the center of mass of the drop at the origin, a "heart-like" shape should be obtained, which qualitatively is a combination of the previous 2 shapes. We thus argue that, the solutions to mode $n = 1$ can be thought of as a combination of the shapes of Figure 2.9 (qualitatively). Generalizing equation (2.22) to 3D (cylindrical and spherical coordinates), the same reasoning as above can be applied. In other words, we expect there to be deformed solutions of a cylinder and a 3D drop whose cross-sections will look like the shapes in Figure 2.5. As such these solutions correspond to the deformation of their boundary when a jet (or drop) is pushed or pulled upon (or both). Naturally, it is also possible in the 3D case to feel a force which is uniformly exerted on the bulk and the boundary (for example a gravitational pull). And in that case, the boundary should be expected not to deform.

Let us proceed by giving some examples of where these shapes seem to indeed manifest, acknowledging that indeed the following examples manifest in 3D. A good example of a "cusp-like" shape is the shape of a Leidenfrost droplet levitated by an air cushion, which shall be discussed later in Part (III). An example of a "drop-like" shape would be a standard drop hanging from a tap faucet before detaching, or as observed in Figure 2.10, a drop running down an inclined plane:
Figure 2.10: “Different shapes of a drop running down a plate when increasing velocity (by increasing inclination). Drops flow downwards. (a), (b) Rounded drops at low speed, (c)–(e) corner drops becoming sharper as velocity increases, (e) corner angle of 60° just before transition to pearling drops, (f) first stage of the pearling drop regime.” [45]

In fact, an example of a “cusp-like”- and “drop-like” shape were found in our experiments, shortly after a pinch-off event. These can be observed in the rightmost snapshot of Figure 1.20. However, the “cusp-like” shape does not have rotational symmetry, and seems to be stretched, which corresponds to a superimposed deformation of mode $n = 2$. As argued in that section, this is to be expected, as the motion of chain-oscillations is propagated into the resulting droplets. Regarding the heart-shapes, Figure 2.8 clearly displays a heart-shape of a drop (with rotational symmetry) at the moment of pinch-off. We were thus interested, whether heart-shapes can also be found near pinch-off in our experiments with chain-oscillations. Magnifying the pictures taken for our experiments, yielded the following snapshots:

Figure 2.11: Snapshots of oscillating drops after breakup in chain-oscillations.

“Heart-like” shapes, a combination of “cusp-like” and “drop-like” $n = 1$ modes, were postulated half-way
through this work. Here we can observe them, captured in time, and presented both in a rotationally symmetric way and in a symmetry broken way, which corresponds to a combination of modes \( n = 1 \) and \( n = 2 \) in \( \theta \) and \( \phi \) respectively (in spherical coordinates). Note, that in the case of a 3D drop the shapes of a mode can be either flattened (in one dimension) or have rotational symmetry. This relates to the harmonic expansion in spherical coordinates, which contains (qualitatively) the same modes as in 2D in \( \phi \) but with the possibility of additional deformations in \( \theta \). As argued before, the oscillatory motion of chain-oscillations seems to propagate into the produced drops. As such, in Figure 2.11 we find some examples of flattened drops, which correspond to a superposition of a heart-like shape with a mode \( n = 2 \), that is, a chain-oscillating heart-shape of a drop. Heart-shapes do indeed manifest in nature very naturally! So does the Cusp-shape and its partner, the Drop-shape. Here “natural” means that these shapes are normal modes of deformation. At this point, we want to proclaim:

We have shown that drops deform naturally into cusp-shapes, drop-shapes and heart-shapes, respectively, when pushed or pulled, or a combination of both!

### 2.4.4 Building on Rayleigh’s work

Most of our additions to Rayleigh’s work, [1], were already mentioned previously: it was noticed that there are (at least) 2 shapes of mode \( n = 1 \) that do deform the drop’s surface. And the association of the normal modes to the shapes of minimal surface area per volume for any geometric “action” was also not discussed (or perimeter per area in 2D). In other words, neither Rayleigh, nor Bohr acknowledged that these modes correspond to the most compact shapes for a certain geometry or effect. Another additional consideration is the identification of a maximum rupture pressure (or energy), related to the mode amplitude. From the linear solution presented in (2.11), we see that the amplitude of the pressure distribution of a given mode \( A_n \) relates to the amplitude of the deformed shape \( \varepsilon = \frac{b}{a} \), namely:

\[
\begin{align*}
R(\phi, t) &\approx a \left[ 1 + \varepsilon \cos (\omega_n t) \cos (n\phi) \right] \\
p(r, \phi, t) &= \frac{\gamma}{a} + A_n \gamma n \cos (\omega_n t) \cos (n\phi) \\
A_n &= \frac{(n^2 - 1)}{a} \varepsilon^{n+1} 
\end{align*}
\]

We now note that \( \varepsilon \leq 1 \), which motivates a finite maximum amplitude (in pressure) for the drop to break:

\[
A_n^{\text{max}} = \frac{(n^2 - 1)}{a} \varepsilon^{n+1} .
\]

In other words, for pressures above the maximum value, a drop will rupture. The pressure distribution (or the velocity profile) can then be integrated over the region that defines the drop, to yield a maximum energy per mode. If we include the term \(-\frac{\varepsilon^2}{a^2}\), this amplitude suffers a slight correction. As the function of the boundary shape is generalized to larger amplitudes, additional corrections have to be taken into account. Still, the condition \( R(\phi) = 0 \) will always impose a critical amplitude \( \varepsilon^{\text{max}} \), which when related to the amplitude in pressure \( A_n \), yields a finite rupture pressure. Therefore — even though the approximations we made break down at high amplitudes and a more detailed calculation is needed — the theory seems to predict that a finite amplitude of a normal mode can rupture a drop! In other words, if a mode is excited with more than that critical energy (amplitude), then the drop can break. And the given form of \( A_n^{\text{max}} \) thus presents a good 1st order estimate thereof, from the linear theory.

### 2.4.5 Building on Bohr’s work

Going through Bohr’s calculations, the only step which left room for doubt was his Taylor-expansion of the curvature operator. Regardless, the operator \( \kappa(\phi) \) can be expanded in full generality, yielding:

\[
\begin{align*}
\kappa(\phi) &= \kappa \left[ R(\phi) \right] = \frac{R^2 + 2\dot{R}^2 - \ddot{R}}{[R^2 + \dot{R}^2]^{\frac{3}{2}}} = \frac{R^2 + 2\dot{R}^2 - \ddot{R}}{[R^2 + \dot{R}^2]^{\frac{3}{2}}} , \\
\left[R^2 + \dot{R}^2\right]^{\frac{3}{2}} \rightarrow (a + \delta R)^{2} + \delta R^2 + [a + \delta R]^2 - \frac{3}{2}
\end{align*}
\]

where we used \( R(\phi) = a + \delta R(\phi) \). Taylor-expanding twice yields:
Continuing with the determination of the radius of convergence, we get 2 conditions to be satisfied for this series to converge onto finite values: it converges onto a finite value, except for $|\delta R(\phi)| \geq a$ (which is a null radius locally) and except for $|\delta R(\phi)| \geq a + \delta R(\phi)$ (which is expected to happen at kinks). This suggests that continuing the perturbation Ansatz from Bohr, the non-linear expansion for the curvature in 2D would behave nicely. This is very important, as it means: even if we cannot solve for the modes at arbitrarily high amplitudes exactly, we should be able to continuously solve the problem at arbitrary order in a Taylor-expansion. Thus, we can know the boundary shapes of the modes at arbitrary order in their Fourier-expansion. In other words, we can perturbatively solve for higher orders, obtaining increasingly more accurate solutions. As these shapes comprise the most compact geometries (because of their connection to surface tension), this could lead to a large amount of applications (and possibly implications). Note that the second condition can be rewritten, and imposing finite curvature now (instead of infinite), we get:

$$|\dot{R}(\phi)| < R(\phi) \implies \left| \partial_\phi \left[ \log R(\phi) \right] \right| < 1.$$  \hspace{1cm} (2.28)

### 2.4.6 Building on other references

Halfway through this work, we found that in recent studies, chain-oscillations are also referred to as "the Axis-switching phenomenon", and their implications on the stability of jets was investigated experimentally by [3, 4]. Using the terminology of [20], separating the breakup of a liquid capillary jet into a "Rayleigh regime" and a "1st wind" regime, they find that chain-oscillations are as stable (or a bit more stable) than circularly symmetric perturbations in the "Rayleigh" regime, and more unstable in the "1st wind" regime. This makes sense, since Rayleigh showed how these modes should be stable for infinitely small amplitudes. However, they have a larger surface area than a circularly-symmetric perturbation, which suggests that they should be more susceptible to breakup in the "1st wind" regime. Nevertheless, a formal theoretical model for this observation is still lacking.

In addition, the mentioned references on this, seem to not understand why a rectangular orifice with aspect ratio of 1 (a square) cannot produce chain-oscillations, while all other rectangles can. This now seems fairly obvious, according to the above reasoning: the mode $n = 2$ is the only one that can stretch the circular shape. A square is not stretched, and therefore is comprised of multiples of mode $n = 4$ (given its symmetry). But a rectangle is stretched, and therefore also holds a large component of $n = 2$. On another note, one can see from (A.6) how $\frac{\alpha}{\alpha_r} = k_r \propto \left[ 1 - \sqrt{2}n(n-1)^2 \left( \frac{1}{Re} \right)^{2} \right]$, where $Re$ corresponds to the Reynolds number. Thus, for $Re \ll \frac{1}{|\alpha_r|}$, the oscillations can be severely damped. This suggests a critical Reynolds number, rather than the critical Weber number discussed in [4].

### 2.5 Understanding chain-oscillations

Now that we have taken a look at the theory surrounding the normal modes of capillary liquids, we can define chain-oscillations, understand their traits and compare the theory to our earlier observations.

Let us begin by discussing the peculiar geometry of the liquid chain. In Figure 2.3 one can see that the oval cross-sectional shape of the liquid shape is not an ellipse. One could argue that the cross-section of chain-oscillations degenerates into the ellipse for infinitely small amplitudes, as the Fourier expansion of the ellipse approaches the boundary shape of a pure sinusoid. In essence this shows how for very small elongations an elliptical orifice produces seemingly "pure" chain-oscillations, which justifies why it has been used in previous investigations. Indeed, one can observe in Figure 1.7, how the surface of the liquid chain becomes increasingly irregular for increasing flow-rates (and thus, amplitudes). This mounts up to saying that the additional components that define an ellipse are negligible at small eccentricities (amplitudes). Still, those
additional components will be present in the respective proportions that define an elliptical shape, which do not match those of the liquid chain’s cross-section. Alternatively, it is plain to see that (2.9) with \( n = 2 \) — which defines the cross-sectional shape — does not define an ellipse (in the case where the curvature in \( z \) becomes important, this equation is altered but it still does not define an ellipse). From (2.8) we can also see how the curvature on the compressed sides of mode \( n = 2 \) becomes negative for a deformation amplitude:

\[
A_2 \geq \frac{\gamma}{a^3} \quad \Rightarrow \quad \frac{b}{a} \geq \frac{1}{3},
\]

where we used \( A_2 = \frac{3\pi}{2\eta a} \) from the solution of (2.11). This implies that at larger elongations, the oval cross-section of the chain-figure deforms inward and produces a "peanut"-like shape, as seen in Figure 1.2. Again, this is very different from an elliptical shape, whose curvature is always positive. Still, it explains why at larger amplitudes the liquid chain displays rounder, thicker edges, as observed in Figure 1.15.

Given the setup of our experiments, “pure” chain-oscillations can only be excited with the ideal choice of the orifice. The shape of a 2D chain-oscillation that we determined with equation (2.8) should produce much purer vibrations than an elliptical orifice. In fact, this should be true for any mode \( n \geq 2 \), using equation (2.9), which should be able to excite higher order “chain-oscillations”, that is, normal modes of a capillary jet given any \( n \geq 3 \). These can be thought to correspond to the cross-section of a jet oscillating as a mode \( n \) of a 2D drop at large wavelengths. However, these will still not be the ideal orifices: there are additional corrections due to the curvature in the \( z \)-axis, and due to the initial velocity excited at the orifice. And because these corrections depend on the flow-rate and the properties of the liquid, the best approach should still be to use the solutions to equation (2.9).

Chain-oscillations are to be understood as a normal mode of vibration of a (cylindrical) capillary jet in 3D. As we have seen, at low energies these modes manifest as harmonic perturbations, as seen in (2.4). With “harmonic perturbation” it is meant here, that the pressure profile of the perturbation is harmonic (it satisfies the Harmonic or Laplacian Equation). As such, we shall refer to these modes as “harmonic modes”, when this is the case. Equation (2.8) presents a good definition of these modes: the shape and dynamics of "pure" modes arise from imposing a pressure profile in the bulk of the fluid, that corresponds to a single term of the harmonic expansion. This presents a simple and intuitive definition of a "pure" mode, and thus for chain-oscillations, which correspond to the mode \( n = 2 \). However, as the equation only holds under specific conditions, there are some additional considerations to be taken into account.

Since the Young-Laplace Equation is a non-linear boundary condition, these harmonic modes only behave linearly for small deformations. That is, only up to deformations where the Young-Laplace pressure drop behaves as a linear boundary, are these really normal modes: under these conditions, the deformations of the boundary behave linearly, so that adding up different components of pure modes one can in theory produce any boundary shape. Conversely, any arbitrarily deformed boundary can be thought of as being the linear sum of their respective normal mode components. That is, the linear effect of the perturbations in pressure is accompanied by a linear response of the geometry of the boundary.

For larger deformations the action of surface tension at the boundary is not linear, and the sum of these normal modes ceases to be linear in their geometry. That is, the pressure profile in the bulk still represents a linear superposition of harmonic pressure profiles, but the shape of the superimposed pressure profiles no longer corresponds to the linear sum of the deformations at the boundary. The geometry of the boundary then needs to be determined by matching the internal pressure distribution with the Young Laplace Equation at the boundary. Still, the geometry will again suffer quantitative changes, but qualitatively display a very similar geometry, as expected from the linear sum.

However, the shape and dynamics of these modes — even pure modes — suffer corrections due to the outer phase, the viscosity and when higher energies/deformations are reached. This can be seen through the Free Surface Condition: at low energies the Free Surface Condition degenerates into the Young-Laplace Equation; but in reality there are additional stresses like the viscous stress, the momentum of the liquid and the pressure of the outer phase:

\[
\left[ \Delta p - 2\eta \frac{\partial u_r}{\partial r} - \gamma K \right]_{r=R(\phi,t)} = 0 ,
\]

where \( \Delta p = p_{in} - p_{out} \) denotes the pressure difference between the inner and outer phases. As convection becomes increasingly important, the pressure can no longer be assumed harmonic. Still, for an incompressible fluid, a potential flow solution exists, for which the potential is harmonic.
\[
\begin{align*}
\vec{u} &= \vec{\nabla} \Phi \\
(\vec{\nabla} \cdot \vec{u}) &= 0 \quad \Rightarrow \quad \nabla^2 \Phi = 0 
\end{align*}
\]  \tag{2.31}

The pressure-profile can then be thought of as suffering additional corrections using the Bernoulli Equation:

\[ p = -\rho \frac{\partial_t \Phi}{2} - \frac{\rho}{2} |\nabla \Phi|^2 + C(t) \quad , \tag{2.32} \]

where \( C(t) \) is a constant that can depend on time. As can be seen from the last three equations, these corrections vanish when \( \vec{u} = 0 \). That is, the corrections arise only when there is movement. Let us imagine deforming the boundary of a jet (or drop) into one mode, to excite the oscillations. As we hold on to the static deformed shape, the pressure inside the fluid would be harmonic, and the corrections would only manifest as soon as we "let go" and the movement of the fluid starts. Although this is not very surprising — the Laplacian Equation trivially describes the static equilibrium dynamics — this represents a natural way to define these modes. The harmonic mode decomposition is exact for static deformations, that is, the normal modes can be thought of, foremost, as normal modes of deformation. This still holds for arbitrarily large deformations, although it should be understood from the point of view of the pressure distribution in the bulk, as we have argued that for large amplitudes the deformation of the boundary ceases to behave linearly.

Then in turn, the modes suffer corrections as soon as movement sets in. For low viscosity and low Reynolds number, the movement is still (arbitrarily close) to the harmonic modes. As we start deviating from these conditions, these harmonic modes can be thought of as suffering corrections up to a regime where they are no longer harmonic. Notwithstanding, as this regime is approached, the normal modes will start to become unstable. As shown in Appendix A, when convection becomes important, the harmonic pressure profile no longer holds and generalizes to:

\[ \nabla^2 \left( p + \frac{1}{2} \rho u^2 \right) = 0 \quad . \tag{2.33} \]

Therefore, the effect of convection will be more noticeable at the extrema of the oscillating “star-shapes”. This can be seen from (2.33): the harmonic pressure profile will be perturbed more at the points where the velocity is highest — ergo, the shape’s local extrema — and it will barely be noticeable near the nodal points. Since some parts of the liquid will feel the effects of convection (which should locally increase the period of oscillation), while other parts will not, the oscillations will dephase and no longer be able to maintain a uniform and coherent oscillation.

In contrast to the liquid chain, it is also possible to observe a liquid jet display a spiral geometry. At first glance, these might look very similar, which is why they can often be confused, or seen as one and the same phenomenon. Nonetheless, they are distinct phenomena and so are their geometrical shapes, as shown in Figure 2.12.

Chain-oscillations represent oscillations of the jet’s surface, with alternating positive and negative elongations (which geometrically represent the 90° turns of the links of a chain). As such, they oscillate through the equilibrium shape (circular cross-section), where there are no elongations. Conversely, a liquid spiral corresponds to a constant elongation of the jet’s cross-section that is continuously rotating around the jet axis. In this case, there are no circular nodal points. That is, a liquid spiral is excited when a jet carries angular momentum (or vorticity) around the cylindrical axis, whereas chain-oscillations possess no angular speed.

The angular momentum can be introduced beforehand as in [21], but it should also be possible to excite the spiral with the orifice alone; for instance an elongated asymmetric orifice, such that surface tension pulls stronger one way around than the other. As an analogy, chain-oscillations can be though to correspond to an oscillating spring, whilst the liquid spiral is equivalent to swinging a spring around in a circular motion, such that the spring is forced to elongate due to the centrifugal force. At a constant angular speed, the elongated spring can now be thought to have a new equilibrium point, around which it can still oscillate, while rotating. In the same manner, it should also be possible to have chain-oscillations superimposed on a liquid spiral. A “chain-oscillating-spiral jet” should thus, in theory, also exist as a combination of both effects; even though it will be quite difficult to excite and to observe.

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Likewise, chain-oscillations are not to be confused with the Rayleigh-Plateau instability. Note in Figure 2.12 that these 2 perturbations can be almost indistinguishable, when observed from a fixed perspective. Recall that the shape of a chain-oscillating jet is given by:

$$R_2(\phi, z) = a + b_2 \cos (2\phi) \cos (kz). \quad (2.34)$$

Thus, for a fixed perspective, the projection of the chain-figure, for instance for $\phi \in \{0, \pi\}$, will amount to:

$$R_2(\phi \in \{0, \pi\}, z) = a + b_2 \cos (kz). \quad (2.35)$$

Noting that the circularly symmetric perturbations associated to the Rayleigh-Plateau instability are described by $R_0(\phi, z) = a + b_0 \cos (kz)$, we can understand why these effects could easily be confused. Although their wavelengths will differ slightly, this will not be enough to distinguish these effects with the naked eye. The difference between them can only be appreciated as one observes the phenomenon from different perspectives. In doing so, the circularly symmetric perturbations will maintain their shape, while chain-oscillations will not, as can be observed in Figure 1.3.
Part III: Polygons in Physics

After finding the shapes of the normal modes of a 2D drop presented in Figure 2.3, we noticed that the same (or very similar) shapes and geometries manifested in a variety of different systems in nature and at very different length-scales. Most of these modes — the normal modes of a drop — resemble polygons which are rounded off at the vertices and in the literature they are often referred to as the "star-shaped" oscillations of a drop (or simply "star-shapes") [9, 10, 11, 12, 13, 14, 15, 16]. Even though this is not the case for the modes \( n = 1 \) and \( n = 2 \), in this part of the investigation we will use the terms polygons and star-shapes interchangeably for the whole set of normal modes. Some examples of physical systems where polygonal patterns manifest, are acknowledged in [16]. However, we found a few more examples, which we would like to present and discuss here. As such, in this last part of our investigation, we acknowledge the emergence of (some) polygonal patterns in nature, and we are interested in understanding if some of them are related, and how. That is, having understood the mechanism and the description underlying the "star-shapes" (or polygons) in liquid oscillations, we want to ask why they appear in different physical systems, and if they are exactly the same shapes, or whether they just look similar.

### 3.1 The emergence of polygons in physics

The examples we present in the upcoming sections constitute a varied and multidisciplinary selection of where polygonal structures can be found in nature. As a consequence, it is definitely not trivial to understand their emergence in full detail in all the given examples. Still, we found 4 justifications that argue for the emergence of these geometries. We will now discuss them, before exploring the examples we give in the following sections.

The purpose of these arguments is to shed some light on why polygonal structures can appear in so many different systems, but above all, they invite the reader to understand why they should appear and where.

#### 3.1.1 The harmonic equation in physics

In analogy to the normal modes of a drop, these modes are expected to manifest for a system which is governed by the harmonic equation. In that case, polygonal shapes can be understood to emerge as harmonic perturbations to a stable physical system. Non-trivially, the Laplacian operator seems to be present in some form or another in (almost) every branch of physics. Depending on the theory, there are certain circumstances or conditions, under which the Laplacian term dominates the dynamics. There the shapes of these modes can manifest. Moreover, as argued in Appendix C, the qualitative geometry and dynamics of these shapes should also emerge in solutions of the Poisson equation.

In fact, the Laplacian- and Poisson equation, describe the (exact) general solutions to many problems in (classical) physics. Many modern physical Theories are mainly generalizations thereof, which very often contain the Laplacian operator. As such, we expect polygonal shapes to manifest in systems where a Laplacian term is predominant in the system’s description and for perturbations that carry an \( n \)-fold symmetry. Note that in the case that there is a general expansion as a solution for the theory — as is the case of the Laplacian- and Poisson equation — these modes represent a (normal-) mode expansion. This implies that all these shapes taken together describe everything that the system is able to do. And, vice-versa, anything that the system is able to display can in theory be captured by the set of all the shapes of these modes.

#### 3.1.2 Geometry and typical requirements

Let us suppose to have a stable physical system, whose equilibrium shape is a circle (the same argument holds for a sphere or a cylinder). If this system is now perturbed, we can argue that there should be a boundary condition whose effect will be to restore the system back to its equilibrium shape. That boundary condition can be required to exist, otherwise the system would not be stable. For small (or low energy) perturbations, we expect the boundary condition to hold the shape together, meaning that its boundary should not break. Thus, for small deformations, we can describe most boundaries to deform sinusoidally, or as a Fourier-sum proportional to \( \exp\{\text{in} \phi \} \). As shown in Appendix C, this is indeed the case when the bulk of the system is governed by the Laplacian- or the Poisson equation. In the case of small deformations, both the perturbations in the bulk and the boundary conditions can be linearized, yielding precisely the same shapes as in Figure 2.3; that is, the normal modes of a drop in the regime where surface tension can be assumed a linear boundary condition (for small amplitudes). Note that in this linear approximation the boundary function is scale invariant and there is thus no requirement for the scale of the system. And as shown in Part (II), section 2.3, the pressure
distribution and flow-profile are also scale-invariant, which should also hold for many hydrodynamic systems where these quantities are included.

As the magnitude of the deformations are increased, moving away from the linear regime, the produced shapes will also start to deviate from the harmonic modes of a drop. Their exact shape will then depend on the non-linear details of the bulk-theory and the associated boundary condition, which in general will introduce scale-variance. However, the shapes are expected to still look very similar, as they should maintain their symmetry. We thus expect that the exact shapes of the normal modes of a drop can be found in other systems in a small amplitude regime. And that very similar shapes should be found away from that regime, or in more exotic systems. As a consequence, we expect the polygonal shapes of the normal modes of a drop to qualitatively manifest in many more physical systems than just a capillary drop. Moreover, the condition that the perturbations on the boundary close down on themselves is generically a requirement for normal modes of oscillation of a given system. As such, we expect these shapes to emerge as the normal modes of a variety of systems with circular geometries.

3.1.3 The emergence of hydrodynamics

There is a more modern and elegant way to understand the emergence of many examples of these polygonal structures, which relates to the latter two arguments: hydrodynamics, the study of how fluids behave, has been increasingly found its laws to emerge in a variety of different physical systems. Examples include, but are not limited to the subjects of elasticity, superfluidity, charge-diffusion, heat-diffusion, magnetohydrodynamics, electron-oscillations in quantum matter, and even gravity [41]. In the last decade, hydrodynamics has increasingly been found to describe low energy perturbations of systems at finite temperatures [42, 43]. To put it another way, fluid dynamics emerges as the description of excitations of systems that are away (but near) thermodynamic equilibrium. As such, hydrodynamics is an emergent theory, which describes a variety of continuous media. Accordingly, one can argue that for any such medium, there will be conditions for it to be well described by harmonic perturbations. In fact, continuous media are generally described by a pressure distribution, which obeys \( \nabla^2 p = 0 \) in equilibrium. For small enough excitations, \( \nabla^2 p = 0 \) should still describe most generic systems quite well, which implies that those excitations are harmonic (or very close to harmonic). Be that system stable, with a circular symmetry (spherical or cylindrical), then there is a boundary condition, and these polygonal shapes are expected to emerge.

3.1.4 The compactness of a system

As argued in Part (II) (section 2.4.1), the normal modes of a drop represent the most compact way to deform a drop (or a circle, cylinder or sphere) into a shape with a particular symmetry (when surface tension is taken as the only boundary condition). For instance, the mode \( n = 2 \), which is associated to chain-oscillations, should represent the most compact way to stretch a circle with a fixed area. Consequently, these polygonal shapes are expected to manifest in situations where compactness is a major requirement and a given symmetry is demanded. In general, most systems that rely heavily on compactness will carry additional requirements, which will introduce additional corrections on these shapes. This means that in the general case, the shapes will only qualitatively be the same as the normal modes of a drop. Conversely, one can also argue that the emergence of similar shapes in a particular system should be related to the requirement of compactness in that system.

3.2 Examples of polygonal structures in physics

In the sections below, we present a selection of examples in which these shapes (or very similar ones) can be observed. This selection is certainly not complete, as we expect these shapes to be able to manifest in a much larger variety of systems and examples. Here it starts getting interesting because, as it seems, these shapes have been hiding in all the phases of matter and most physical theories all along. The exposed theoretical connections could thus have widespread implications. Even though the theoretical connection between these examples is not always clear or trivial, we have found that the link between the geometry of a given mode and their effect for the case of a drop generalize to all the examples we found. Hence, we believe foremost, that the normal modes of a drop can be a powerful tool to understand complicated physical problems qualitatively as a first approximation. This means that the geometries of the normal modes of a drop can provide insight into the generalised systems; or one could call it “intuition” for physical phenomena, given their geometries (symmetries).
Acknowledging the properties of the harmonic solutions, we are equipped to understand arguably many of the following physical phenomena, at least intuitively. Adding to this, we can think about extra non-linear terms, which might arise due to non-trivial boundary conditions, or other intricate phenomena. Thinking about these extra contributions, one will (at least for the most part) be able to categorize them, using (in)stability arguments. These will ultimately determine, whether the extra contributions produce only mild corrections, chaotic behavior, singularities, phase transitions, or other types of non-linear behavior.

### 3.2.1 Droplet dynamics

According to what has been presented so far, droplet dynamics shows the harmonic mode's geometry trivially, by definition. Although it does so in spherical coordinates, whose boundary mode shapes correspond to Spherical harmonics at small amplitudes (in the linear theory), we have argued previously, that often they seem to degenerate into the 2D polar modes. Here we present 3 examples thereof:

![Figure 3.1: Leidenfrost drops and Leidenfrost puddles (respectively) [14, 15].](image)

The spheroidal Leidenfrost drops and Leidenfrost puddles seem to conform quite naturally to the geometry of the harmonic modes of pressure for a 2D drop in polar coordinates. Similarly, a drop levitated by sound-waves will display similar behavior, [10], which marks the third example. In these examples, drops are parametrically excited to oscillate, and at the resonant frequencies of these modes, their geometry is revealed. We strongly encourage watching the videos of the drops oscillating in time, presented by these references. Here one can see a drop oscillating on a plate in mode 2, and a sound-levitated drop oscillating in various modes. On another interesting note, a recent paper [19] proposes to calculate the normal modes of a 3D drop in spheroidal coordinates, using Riemannian geometry.

### 3.2.2 Bubble dynamics

Consider there may be 2 different phases of the drop (actually 3 with the outer phase of the surrounding air). This case in spherical coordinates is the case of a bubble, which comprises a gaseous sphere in a thin liquid layer around it. According to [46], the standard modes and their associated oscillation frequencies get split into 2: an in-phase frequency, in which the 2 inner phases of the drop oscillate in sync. And an out-of-phase frequency, in which the phases oscillate out of sync (in phase opposition).

### 3.2.3 Rotating hollow fluids

As one can see from the following example, it is also possible to have air as the inner phase and liquid as the outer phase:
Here the system is forced to rotate, otherwise there would not be a hollow interior anymore. Therefore, in this case it is the centrifugal force which excites these modes and surface deformations. Note that the centrifugal force introduces a perturbation in pressure, which has to be taken into account:

$$\nabla^2 p \simeq \rho \omega^2.$$  \hspace{1cm} (3.1)

As such, while these hollow modes can degenerate into harmonic modes when $\rho \omega^2$ is negligible at the liquid air interface, they are really not harmonic anymore. Still, they do manifest and their shapes and behaviour are qualitatively the same as the harmonic modes of a drop. Given that the system is rotating as a whole with a certain angular frequency, it is evident why different modes are excited in the external liquid: the internal and external parts are at different distances from the center (radii), they have a different thickness, and thus they have completely different harmonic frequencies. Consequently, at the same angular frequency, they are expected to excite completely different modes. In fact, they could only match the same mode under very specific conditions, or proportions.

### 3.2.4 Flow around objects and singular corner flow

As captured well in the book Singularities: Formation, Structure and Propagation [5]: in 2D, a Potential Flow in a corner is described by self-similar bi-harmonic functions:

$$\Phi = r^\lambda f(\phi).$$  \hspace{1cm} (3.2)

Notice how the harmonic modes we found are special cases of these functions, which are scale invariant as well. These solutions are known to appear in steady flows around objects / obstacles / sharp corners. In polar coordinates (2D), the flow potential ($\Phi$) and the stream-function ($\Psi$) usually take the form:

$$\begin{cases} 
\Phi = r^\lambda \cos (\lambda \phi) \\
\Psi = r^\lambda \sin (\lambda \phi) \\
\lambda \in \mathbb{R}
\end{cases}.$$  \hspace{1cm} (3.3)

In the case that $\lambda \in \mathbb{Z}$, these are precisely the same functions that describe star-shaped oscillations (or the harmonic modes of pressure). Examples for $\lambda = n \in \{1, 2, 4\}$ are, respectively:
To be more precise, for the examples above corresponding to \( n \in \{2, 4\} \) the potential functions also contain a term \( \Phi \propto v_\infty x \), as the potentials described above correspond only to the perturbations of a constant flow around objects. Seeing the vortices appear in this example, one realizes that near every peak of every mode some vorticity can be excited as it oscillates, but some positive and some negative, so that when it oscillates smoothly they cancel out.

Although there is no vorticity in the linear solution we derived earlier, due to the nature of the oscillations and the produced flow profile, it seems intuitive that such vortex-pairs could be excited at higher energies when convection becomes important. If the oscillation is violent enough, vorticities could arise near the extrema and develop vortex motion. And in theory, if this happened symmetrically, each mode could develop \( n \) equally sized and spaced vortex pairs (so \( 2n \) vortices) with 0 net vorticity. This also holds for \( n = 1 \), which would produce 2 vortices near the "cusp-tip".

### 3.2.5 Shockwaves

Shockwaves often display similar shapes, which can easily become singular. Here we present 2 examples of previous works where such harmonic shapes seem to manifest:
Generalizing our analysis to a compressible fluid, we have:

$$\partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0$$

$$\Rightarrow (\nabla \cdot \vec{u}) = -\partial_t \log \rho - \frac{1}{\rho} \nabla \rho \cdot \vec{u} \quad .$$

(3.4)

From this, and assuming \( \vec{\omega} = \nabla \times \vec{u} = 0 \), we obtain:

$$\nabla^2 (\rho + \frac{1}{2} \rho u^2) = [-\partial_t + (\zeta + \frac{4\eta}{3}) \nabla^2] (\nabla \cdot \vec{u})$$

$$\Rightarrow \nabla^2 (p + \frac{1}{2} \rho u^2) = [\partial_t - (\zeta + \frac{4\eta}{3}) \nabla^2] (\partial_t \log \rho + \frac{1}{\rho} \nabla \rho \cdot \vec{u}) \quad .$$

(3.5)

In shockwave dynamics, these shapes indubitably exhibit very rich, non-linear dynamics. Still, it can be argued that for some conditions \( \nabla^2 (p + \frac{1}{2} \rho u^2) = 0 \) is valid on the shock. Perhaps even \( \nabla^2 p = 0 \) for small velocities in the early phase of the development of the shock. This would certainly be congruent with the last two examples in which the harmonic shapes seem to manifest at an early stage of the shockwave.
3.2.6 Mechanics and Elasticity Theory

The buckling behaviour of metal cylinders can be decomposed into normal modes of buckling under GBT (General Beam Theory), [25, 26]. And these seem to be precisely the shapes of the normal modes of a liquid cylinder, including the shape of chain-oscillations (in the 2D polar-cross-section approximation).

Figure 3.6: Figures taken from [25]: “Generalised beam theory to analyse the buckling behaviour of circular cylindrical shells and tubes” - IST, Lisbon.

Note that in the case of chain-oscillations, there is a fixed cross-sectional area and the perimeter is variable, while for the case of a buckling metal cylinder, the opposite is true. Still, the same geometries emerge, at least qualitatively. We thus see how a chain-figure seems to be the most natural deformation of a cylindrical system, if that system is stable enough to display it.

Similarly, the normal modes of “deformation”, or the normal modes of vibration of an elastic circular solid should be the same as in a liquid (for small amplitudes and the right approximations). However, away from those approximations, a qualitatively similar behaviour is still expected. This also holds in other coordinates and dimensions, for example in cylindrical and spherical coordinate systems.

3.2.7 Cracks

Polygonal patterns have also been reported to emerge in cracks, [16]. This might be due to the biharmonic equation, which underlies the phenomenon.

\[ \nabla^4 \Phi = \Delta^2 \Phi = 0 \]  \hspace{1cm} (3.6)

It is thus no surprise that crack patterns can display similar polygonal structures. Nevertheless, it would be interesting to study whether there are certain conditions, for which cracks will naturally tend to display these polygonal shapes.

3.2.8 Electromagnetism

Maxwell’s Equations are no exception to the Laplacian operator. A good example to think about is Magneto- and Electrostatics, which are governed by the Laplacian and Poisson Equation, respectively:

\[
\begin{align*}
\nabla \cdot \vec{D} &= \rho_e \\
\vec{D} &= \varepsilon_0 \varepsilon_r \nabla \Phi_e
\end{align*}
\]  \hspace{1cm} (3.7)

Indeed, the electric potential around charges, in empty space, obeys \( \nabla^2 \Phi = 0 \). The electric field lines represent nothing but the gradient of the harmonic potential that satisfies the boundary conditions. Meanwhile, it appears that the 2D mode shapes are traced out by the harmonic functions or their harmonic conjugate functions, depending on whether the force is attractive or repulsive.
Figure 3.7: 2 electric charges. **Left:** a dipole. **Right:** 2 repelling charges.

This example clearly illustrates how the harmonic conjugate function swaps with the harmonic function, when we swap the sign of one of the charges. Observing the field lines, one can also see how the shapes of the field lines on the left are the same as the equipotential contours on the right and vice versa. Interestingly, Electrostatics contains the same functions and shapes as droplet dynamics. And at a time, where we were not sure by which function the drop-like shape is described, or even why it was there at all, this duality with Electrostatic fields helped us understand. Namely, it seems to suggest that the "drop-like" and "cusp-like" shapes presented earlier could be associated to harmonic conjugate functions. Some more examples of distributions of point-charges follow:

Figure 3.8: Symmetric distributions of 4 point-charges.

Note that if the distribution of charges is symmetric, we expect the emergence of the shapes of a "pure" harmonic mode. This is why in the latter figures we can observe the emergence of a harmonic shape equivalent to $n = 4$. 
In Figure 3.9 on the most left, we see the equivalent of an $H_2O$ molecule being attracted to a charge of $-3e$. The other 2 fields correspond to the pairs $(e,-3e)$ & $(e,-2e)$, respectively and they clearly display "heart-like" shapes.

### 3.2.9 Gravity

Similarly, Newtonian gravity also obeys the Poisson-Equation and is equivalent to 2 charges of opposite sign, since it is attractive:

$$\nabla^2 \Phi_G = 4\pi G \rho.$$  \hspace{1cm} (3.8)

The interesting question is how this relates to general relativity. Vitor Cardoso, a portuguese physicist, (with multiple awards including NASA and the American Physical Society) believes that understanding these shapes and dynamics are a crucial first step towards understanding the dynamic merging process of black holes [37]. As a matter of fact, the study of bubbles and biophysical membranes has been increasingly used in the description of black holes, black branes and black strings [38, 39, 40]. This gives an additional motivation to the importance of the study of the normal modes of drops (and bubbles) for advanced topics in theoretical physics.

### 3.2.10 The heat Equation

At this point, it should be obvious that the solution-space for the time independent Heat-Equation (and many static equilibrium problems) should be spanned by the harmonic expansion. Namely, it obeys the harmonic equation:

$$\nabla^2 T(\vec{r}, t) = 0.$$
3.2.11 Standing waves

Wave interference and wave-front propagation can display polygonal shapes at short instances in time. Good examples for this are cymatic shapes, or in other words, the normal modes of vibration of a surface, for instance, liquids in circular containers:

![Pentagonal cymatic mode in a circular container with water.](image)

Figure 3.11: Pentagonal cymatic mode in a circular container with water.

The dynamics here can get quite complicated. However, looking at the wave-equation, \( \nabla^2 \Phi = \frac{1}{c^2} \partial^2_t \Phi \), understanding this might be as simple as associating the shapes to the instants in time where:

\[
\partial_t^2 \Phi \simeq 0 \implies \nabla^2 \Phi \simeq 0.
\]

Note that for normal modes of oscillation this corresponds to the nodal points (or nodal lines). In short, a “cymatic mode” with a symmetry of \( \frac{2\pi}{n} \) can produce a shape, which looks like the corresponding “star-shaped” mode \( n \) in polar coordinates (or an integer multiple). As a neat example, this implies that these polygonal modes should be contained in the patterns observed in Chladni-plates. And this seems indeed to be the case:

![Triangular Chladni-plate patterns.](image)

Figure 3.12: Triangular Chladni-plate patterns [27].
In fact, these shapes can be present in Faraday-waves (non-linear surface oscillations) as well. Faraday waves displaying similar shapes on a circular water surface are studied in [28]. Likewise, they can be seen to emerge in liquid metal as well, as studied in [29].

3.2.12 Bose-Einstein Condensates

It is known, that a BEC in the Thomas-Fermi-regime displays superfluidic properties. Be this the case, polygonal modes have been reported in circular and torus-shaped trapped BEC’s, as well as around vortex-systems in BEC’s [30, 31, 32, 33].
Here, it should be stressed that the dynamic boundary of a BEC is subject to different boundary conditions, than in the elasto-capillary case of water. Interestingly, the boundary condition in the BEC case allows for sharp corners very naturally, producing “normal” polygons with sharp vertices.

3.2.13 2D Vortices

Let us start by thinking about why and where these shapes appear in vortex motion, by considering a 2D vortex: it is known that vortices are linearly stable, that they tend to have a circular geometry, and that in real vortices, the vorticity is “smeared out” over the “eye” of a hurricane, as opposed to just existing in its center point. Recall equation (A.11):

$$\nabla^2 \left( p + \frac{1}{2} \rho u^2 \right) + \rho \left[ \mathbf{\tilde{u}} \cdot (\nabla \times \mathbf{\omega}) - \omega^2 \right] = \left( \nabla \cdot \mathbf{f} \right) .$$

(3.10)

Note now, that at the boundary of the eye of the vortex (and further out) $\mathbf{\omega} = 0$. Putting also $(\nabla \cdot \mathbf{f}) = 0$, we get:

$$\nabla^2 \left( p + \frac{1}{2} \rho u^2 \right) = 0 .$$

(3.11)

For an ideal vortex $\mathbf{\tilde{u}} = \nabla \Phi = \frac{\Gamma}{2\pi r} \mathbf{e}_\theta$, so that:

$$\nabla^2 p \simeq -\nabla^2 \left( \frac{\rho \Gamma^2}{8\pi^2 r^2} \right) .$$

(3.12)

Usually, $\nabla^2 \left( \frac{\rho \Gamma^2}{8\pi^2 r^2} \right)$ should be small from the eye outwards, so that $\nabla^2 p \simeq 0$. Either way, if there is a perturbation on the flow, that locally creates a pressure which is greater or comparable to $\frac{\rho \Gamma^2}{8\pi^2 r^2}$, then to a good degree that pressure satisfies $\nabla^2 p = 0$. We know that at that point there should be a restitutive force (because vortices are stable), which should be linear as a 1st approximation. Thus, we have a stable circular shape, whose disturbances in pressure must follow the harmonic equation. Furthermore, these perturbations
are described by self-similar functions and are thus scale invariant, just as vortex motion itself!! This is why a circle surrounding the center eye (smeared vorticity) can display exactly the same shapes and pressure profile as the star-shaped drops! This holds up to the point where the boundary condition in the vortex case deviates from the boundary condition provided by surface tension. While it is still unclear what the implications are for a vortex to assume a pure mode, we now understand at least why these star-shapes can appear in vortex motion, and that (for small amplitudes of the perturbations) they are precisely the same shapes! Again, we note that in the case that $\nabla^2 p \neq 0$, similar shapes can manifest, which will in general differ from the harmonic modes, but still very much resemble them.

![Figure 3.18: Polygons on a rotating water surface [34].](image1)

![Figure 3.19: Hexagonal pattern and associated barotropic vortices [35].](image2)

Notice that we can observe the 2D modes from above in an actual (cylindrical) 3D vortex, due to the fact that these problems should be "smoothly" connected, as argued previously. These shapes are observed on the contact lines with the center eye, but they should also be able to emerge as narrow jets which circulate around symmetric vortex-systems. Iconic large-scale examples thereof are the hexagon on Saturn’s north pole, Jupiter’s great red spot and Jupiter’s north pole’s vortex-system. Pictures thereof were taken by NASA and are displayed in the following pages:
Figure 3.20: The hexagonal pattern on Saturn’s north pole. Image taken by NASA.
Figure 3.21: Jupiter’s giant red spot. Image taken by NASA.

Figure 3.22: Infrared picture from Jupiter’s north pole: the central cyclone at the planet’s north pole and the eight cyclones that encircle it. Image taken by NASA.
3.2.14 Vortex rings

Vortex rings seem to be connected to these shapes in a very special way. Later on, we will argue that the formation of a vortex-ring seems to be a more violent form of the deformation of a spherical drop into mode $n = 1$. Furthermore, we just saw how polygonal perturbations can develop around a vortex line, and thus, they could live around a vortex loop, in favorable conditions.

Nevertheless, perturbations to the circular vortex-ring itself also seem to produce these shapes. Again, there is a stable, circular configuration, that can display the star-shaped oscillations. In this particular case, however, the perturbations do not oscillate in 2D, but rotate around the vortex-ring, so that their projections onto 2D look like the star-shaped oscillations. There seem to be no studies on this, and as such, no pictures. However, the author found a video of a simulation where this can be observed:

To see these rotations in mode $n = 3$, please watch the first few seconds of Knocked vortices in real fluids, starting at min 1:00.

The fact that the perturbations do rotate is a consequence of the conservation of angular momentum (since there is rotation around the vortex ring, when it deforms, the upper part of the deformed ring has more angular momentum than the lower part of it, which means there is a net angular momentum to that perturbation). Notwithstanding, the mechanics underlying this precession and their stability still need further investigation.

3.2.15 String Theory

Somehow unsurprisingly, these modes are studied in String Theory, as some of the normal modes of vibration of a closed string. Generalizations of these modes in higher dimensions are also expected to manifest as long wavelength dynamics of black-branes, because in that regime the Laplacian dominates.

3.2.16 Plasma physics

There seem to be reports of polygonal patterns in plasma physics, too; [16, 22]. An interesting variation of these shapes seems to, indeed, be present in Stellarator Fusion-reactors. In fact, they seem to be normal modes of plasma-tori, which twist around $n$ times with the perturbations of a mode $n$:

Starting with an ideal toroidal shape, a plasma will suffer perturbations as it flows around in a circle. This sets a stable operating mode in the case of $n$ symmetric such perturbations in one loop. The twisting of the plasma shapes can be understood as the manifestation of the Lorentz-Force, which is proportional to $\vec{I} \times \vec{B}$ where $\vec{I}$ is the electric current and $\vec{B}$ the magnetic field. This twisting motion seems to be in phase with the local extrema at the boundaries of each respective shape of a mode $n$, which again is a requirement for a stable operating mode of the reactor. As the tori twist around the shape, there is a centrifugal force which stretches the plasma. As such, the cross-section of a stellerator is not circular and in fact looks similar to a mode $n = 2$. As in the case of the vortices and vortex-rings, further and more concrete investigations are needed on these plasma modes, however.
3.3 Normal mode reasoning

In the last section, we showed how the harmonic modes (and very similar shapes) manifest in systems which are governed by the Laplace- and Poisson equation (or variations thereof). In this section, we want to illustrate that understanding these shapes can be a powerful tool in understanding complicated physical systems qualitatively, by using the intuition gained from studying the harmonic mode decomposition. In other words, we just showed that such modes can be found in a variety of systems. And now, we are interested in showing what we can learn or expect, when such shapes are observed. In fact, in the last section on plasma physics we already presented justifications of why a Stellarator Fusion reactor should have the shape it displays, by only using our intuition from the normal modes of a 2D drop.

As such, let us now appreciate our results so far. In particular, please notice how in 2D, $n = 1$ is the only mode that can produce movement about the center of mass, and $n = 2$ the only one that can stretch a shape. This fact holds in the discussed 3D coordinate systems as well.

3.3.1 Leidenfrost-drops

Consider a drop levitating through the Leidenfrost-effect. To describe this system, we need to take heat and temperature into account, which gives rise to many complications, like Marangoni-effects, evaporation etc. Still, we can understand the shape of a Leidenfrost droplet qualitatively with these modes, while ignoring all these complicated effects: gravity acts on all the particles of the liquid uniformly and simultaneously, and does not deform its surface. The drop has to deform its surface, however, to counter gravity. And the only way an air cushion can provide a constant force to a drop from the outside, is by curving its surface. Therefore, in order to support a drop, we need a $n = 1$ deformation, and there are thus 2 shapes to choose from, but one of them minimizes gravity, namely the cusp-like shape. Notice that gravity can be minimized best through the $n = 2$ shape. Since it is the only one stretching, it is the only shape that really lowers the center of mass. As the drop grows bigger, and the weight of the drop becomes more and more important compared to the surface tension forces, it will develop a chimney-instability and rupture in the center, as discussed in [44].

![Figure 3.26: Numerical solutions of the shapes of differently sized Leidenfrost drops, [36].](image)

Notice in Figure 3.26 how the equilibrium shape of Leidenfrost drops for increasingly bigger drops looks remarkably similar to the linear superposition of modes $n = 1$ and $n = 2$. This is in congruence with the arguments above, explaining how a component of the 2nd mode increases for increasing gravitational effects, since the size of the drop is proportional to the weight of the drop.

3.3.2 Swirling jets

Consider swirling jets as in [21] (or rotating drops as in [37]). The only modes that have potential energy (at least at small amplitudes) are the modes $n \geq 2$. A rotating jet (or drop) will thus break its symmetry spontaneously and deform its shape into one of the modes. Nonetheless, the shape of the modes is a bit different, as in this case the centrifugal pressure enters the free surface condition for matching the deformation at the boundary. In other words, there is a centrifugal "source" term, and the Laplacian of the pressure is no longer 0.
Qualitatively, however, the process is quite straightforward to understand from these modes: mode $n = 2$ is the one with less energy per elongation, as it represents the fundamental mode. Thus at low angular velocities, the system will elongate into $n = 2$, just like a spring-system elongates, when swung around. If you raise the angular frequency, the centrifugal force is larger and the elongation too, but at some point, the elongation of mode $n = 2$ would become way too large (and therefore its surface area). We can achieve the same potential energy with a higher mode at less elongation. Thus, the circular jet can spontaneously break its symmetry into any mode, depending on the starting angular frequency. This phenomenology gives rise to the complicated bifurcation diagram of these jet-modes (drop-shapes) and to the helical shapes of the swirling jets observed in [21].

![Figure 3.27: Rotating jets [21]](image1)

![Figure 3.28: Bifurcation diagram [21]](image2)

### 3.3.3 Cells

For a physicist, an elephant is a sphere, and so is a cell (to zeroth order, at least). Of course cells can display a variety of different shapes, for a variety of reasons. Some change their shape actively, while others maintain their shape fairly well over time (usually by use of their cytoskeleton). Irregardless, a cell’s membrane is subject to surface tension. Thus means that if a particular cell-type would like to be a bit stretched but still as compact as possible, the most efficient way to achieve this would be to deform as a pure mode $n = 2$, because it is the only mode that stretches, and because adding a contribution from another mode would only increase the potential energy (surface area). If a cell senses a constant push from its neighbour, it will also display a cusp-like shape. In biology, the correct technical term is to call it an “irregularly contracted cell”. Similarly, a “Dacrocyte” looks a lot like our drop-like shape:

![Figure 3.29: Cell classification](image3)

![Figure 3.30: Cell](image4)

![Figure 3.31: Cells under microscope](image5)

Let it be noted, that cells are 3-dimensional and not 2. Therefore, the cross-sectional shapes we are seeing are supposed to disagree slightly with our 2D shapes in polar coordinates. As discussed before, this arises since there is an extra dimension of curvature adding to the effect of surface tension, which has to be matched through boundary conditions. Yet, the reality is that they look remarkably similar. Following the previous train of thought, we point to the fact that cell division would be most efficient, if the cell was programmed to self-excite a pure mode $n = 2$. More concretely, in stretching this way, additional surface costs are avoided,
as well as energy dissipation due to unnecessary flow dynamics. Looking under the microscope, we find that indeed there seems to be a remarkable "coincidence" in the dynamics of mitosis:

![Image 1](image1.png)

**Figure 3.32: Cell-division under electron microscope. [54, 55]**

The way the embryonic cells stick together also resembles the harmonic modes and their respective shapes:

![Image 2](image2.png)

**Figure 3.33: Embryonic stages of Lophelia pertusa. [56]**

Further examples of the importance of these shapes in cytobiology can be found in multipolar mitosis:
3.3.4 Fruits and shapes in biology

Many fruits are also quite spherical as a first approximation. But they of course assume varied forms, with different geometries for different purposes. However, the contact between the fruit’s contents and the exterior is unfavorable and is expected to be minimized. Therefore, we expect surface minimization to play an important role. Put differently, we expect that through evolutionary processes plants tended to maximize the compactness of their fruits, while taking other requirements (or restrictions) into account.

Since the normal modes of a drop represent the most compact shapes with a given effect or n-fold symmetry, one could expect similar shapes to be found in fruit. This should hold also for other biological systems, for which compactness is a strong requirement. Here, we can also argue for the movement about the center of mass to be related to the shapes of mode $n = 1$. We know since Newton that apples and pears are subject to gravity, while still holding on to a tree. And just as in the Leidenfrost example, the way to counter or deform in one direction, while still trying to be as compact as possible, will result in rotationally symmetric cusp-like (apple/tomato ...) and drop-like (pear/strawberry) shapes. This can be observed in Figure 3.35 for an apple, a pear and a tomato.

Figure 3.35: Apples and tomatoes have a cusp-like cross-section, while most pears mimic a drop-like cross-section.

As a matter of fact, there are many more such examples, including some of the polygonal shapes as well. Some are displayed in Figure 3.36.
Again, this is because any geometrical symmetry that might give rise to evolutionary advantages, will very often be maximized in terms of compactness. And this is by definition what the modes represent, when taken in the context of surface tension. It is thus not that surprising, that some of these shapes (or very similar ones) are found in biological systems, where compactness is a major factor in an associated cost-function, which has been optimized through uncountable generational cycles and evolutionary pressure.

### 3.3.5 Coffee cup

Playing around with a shallow rest of coffee, tilting the cup just a bit and wiggling it, the modes $n = 1$ can also be seen. Here are some pictures thereof:

Figure 3.36: Several fruits and vegetables assume cross-sections with very similar shapes to the normal modes of a 2D drop as in Figure 2.3.

This can be observed, depending on the shape of the cup. The cup should, ideally, be a half-sphere with a flat bottom, like in the example in Figure 3.37.
3.3.6 Sink-cusp

If it was not obvious by now, why the first mode appeared in the coffee example of the previous section, maybe it will be after this example:

![Picture of a sink-cusp](image)

The corresponding video to this picture is appended as Video 2. We do recommend the reader to try it out at home, as it is much easier to perceive the motion of the fluid. This sink in particular has the shape of a quarter-sphere, attached to a flat wall. If it were a half-sphere, the seemingly static boundary shape of this arguably complex fluid motion would be a circle. Adding the flat wall thus corresponds to a constant force in one direction, which excites mode \( n = 1 \). Meanwhile, the water-jet coming from the drain serves as a source of fluid and momentum, which spreads in a thin sheet on the surface of the sink and equally in all directions (this depends on the incidence angle, of course). This represents the interior part of this system, which in Video 2 appears a sort of peanut-shaped hole in the water. This incoming jet then hits the previously expelled water, which has transformed its kinetic energy into gravitational potential, and is already on its way back down again. This sets the condition for the boundary of this flow (with some fluctuations due to the formation of bubbles).

When the flow-rate of the incoming water compensates the drainage, a dynamical equilibrium can take place, and the boundary shape displays a stationary cusp-like shape. The interesting part, is where the incoming water-flux hits the layer of the receding boundary-water. Since the spreading water sheet is typically much thinner (smaller), a sort of vortex-motion develops, as the thin sheet penetrates from below, forcing the top part of the boundary-water to fall down, and then inwards again due to the momentum at the bottom. Since this is going on everywhere on this interior boundary, this illustrates again, how these shapes combine very naturally with vortex-rings! This system has in essence a vortex ring boundary mechanism, which displays the cusp-like shape of mode \( n = 1 \). In addition, one can notice how 2 stationary and counter-rotating vortices develop near the cusp-tip. This notion is the subject of the next few sections.

3.3.7 The vortices in Sistelo, Portugal

Wandering around in the north of Portugal, in midst of the wild, the author came across a man-made dike along a natural river. In the corner between the dike and the river-bed, this was observed:
A brief video of the surroundings and the rotating structure is appended as Video 3. Naturally, the flux of the river, which presses against the concrete of the dike, exerts some force. At the corner, where the effect was taking place, the corner was already rounded off, and had small holes under it, where the vortices could extend to. As a consequence, a steady $n = 1$ potential flow developed. This can be understood as the equipotential lines around 2 counter-rotating vortices naturally displaying a cusp-like shape, as argued previously in the section about vortices, 3.2.13. Capillary waves on the water surface then make than geometry apparent, as they approach the vortex-pair. Conversely, it seemed pretty clear as the effect was observed, that the region of that vortex-pair was pressing strongly against the dike, which would explain the more round corner of the dike. Again, the cusp-like geometry seems to emerge associated to a net force in one direction, that is a movement “about the center of mass” of that region of fluid. From what we have seen so far, it makes sense that such a steady flow should be allowed. Nonetheless, it begs the question:

Was it the holes, which excited this mode at the corner, leading to the slow erosion process of rounding-off the walls? Or was it the slightly round corner, which produced 2 vortices, which then in turn popped open the holes? So, which came first, the chicken or the egg? A physicist would certainly say: “It really does not matter. As long as both can manifest, they slowly will over time, one way or another.”

### 3.4 Possible generalizations

#### 3.4.1 Vortex generation

Assuming that movement about the center of mass still follows qualitatively the shape/dynamics/behavior of mode $n = 1$ at more violent energies, this mode should appear in the context of a constant force (through a constant jet flow) and be able to produce 2 vortices. In fact, this is very much in agreement with the generation of many vortices in nature:

![Figure 3.40: Illustration on 1st mode](image)

![Figure 3.41: Double-vortex in a pool](image)

![Figure 3.42: Rayleigh-Taylor Instability](image)

When we stir coffee, we see this dynamic in 2D, as if the spoon was pushing on a 2D drop (whose scale is set by the parameters of the system), deforming it and generating the 2 vortices (if we give it a strong push).
The mushroom-cloud formed by a nuclear explosion can be loosely thought of as a sphere of already cooler smoke being pushed by an incoming flow of hot new smoke. Therefore, the head of the mushroom-cloud looks very similar to a 3D drop deforming in a cusp-like shape with axial symmetry. Here, in 3D spherical coordinates, the vortex-pair in 2D now become a closed vortex-ring. Thus, a vortex ring can be thought of as being generated through a more violent deformation of mode \( n = 1 \) (cusp-shape) in 3D spherical coordinates.

A similar geometry and flow are observed in “wake-turbulence” generated by a helicopter as it hovers. When it starts moving forward, this becomes a cylinder of mode \( n = 1 \), whose 2 vortices are now vortex-lines. The same cylinder is seen in air-planes, and is also commonly referred to as “wing-tip vortices”. Again, one can argue that these shapes manifest due to the need of a force about the center of mass, which in this case relates to the lift force.

### 3.4.2 Attractors and fixed lines

The Laplacian operator does not depend on time. Therefore, when looking for fixed points in dynamical systems, or more generally, fixed surfaces, this term always survives. In essence this means that in a dynamical system in physics, a fixed surface is very often defined by a surface that obeys the harmonic equation (Laplace-Equation) or a generalization thereof. Acknowledging this, it is no wonder that similar shapes and dynamics are observed in compressible and turbulent flows.

### 3.4.3 Quantum Field Theories

There are reports of polygonal patterns in Quantum Field Theory (particularly in Condensed Matter physics), which look similar to the harmonic modes, \([16, 51]\). Nevertheless, further investigation is needed, to understand how this extends to these cases.
3.4.4 Optics

Optical modes obey slightly different equations, notwithstanding, some of them look astonishingly similar to the harmonic modes, in particular the modes $0_n$:

![Figure 3.46: Transverse lasing modes](image)

The concrete relationship the shapes of harmonic and optical modes remains yet to be understood.

3.4.5 Atomic and nuclear orbitals

Given that the harmonic modes of a spherical drop at small amplitudes are the spherical harmonics, it would be interesting to ask, how this relates to the orbital-cloud distributions of the atomic orbitals and nuclei. Firstly, in the case of the atomic orbital, we are talking about a quantum-mechanical system. However, in solving for the Eigenfunctions of the Coulomb-Potential-operator in the Born-Oppenheimer approximation, it should be noted, that the Laplacian is intrinsically involved as the kinetic energy operator. Secondly, we believe that similar compactness arguments and symmetries in the nuclear orbitals (or shells) might be at the heart of describing the emergence of the so called Magic numbers of the atomic nuclei, [52, 53]. Take it with a grain of salt, or as a conjecture, if you will.

3.5 References regarding polygonal shapes in physics

To conclude, we composed a list of examples of where these shapes appear throughout different domains of physics (or at least very similar shapes to these). Examples include, but are not limited to:

- Hydrodynamics
  - Droplet dynamics [1, 46]
  - Bubble dynamics [46, 47, 24]
  - Star-shaped oscillations [9, 10, 11, 12, 13, 14, 15, 16]
  - Vortices [34, 35, 22]
  - Oscillations of torus-shaped fluids [49]
  - Vortex rings
  - Shockwaves [5, 18]
  - Rotating drops [37]
  - Rotating hollow fluids [22]
  - Swirling jets [21]
  - Bluff body-flow and stagnation points [23]
• Mechanics
  Buckling of unstiffened cylinders [25, 26]
  Friction Dynamics of vehicle Brake Systems [48]
  Vibrational modes (or modes of deformation)
  Crack patterns [16]
• Waves
  Cymatics [27]
  Faraday Waves [28, 29]
  Frequency Combs [50]
• Plasma physics [22]
• Bose-Einstein Condensates
  Circular-shaped Bose-Einstein Condensates [30, 33]
  Torus-shaped Bose-Einstein Condensates [31]
  Multi-Vortex configurations in rotating Bose-Einstein Condensates [32]
• Condensed Matter Theory [16]
  Distribution of Polyakov loops on the complex plane [51]
• General Relativity
  Understanding these modes can help understanding phenomena related to gravity and Black Holes, according to [37]

**Perspective and Outlook**

In this section we want to point out the most promising observations, which should be explored and generalized in future work. An important next step would be to solve for a singular flow solution and to model the exit angle and the parameters of the jet as a function of the flow-rate \( Q \). This will be of value for modelling and understanding capillary jets for arbitrary orifices. Naturally, the study of non-circular jets should not remain exclusive to \( n = 2 \), as it can be extended for modes \( n \geq 3 \). In this regard, the shapes (and dynamics) of these modes should be solved for higher amplitudes, which can be accomplished numerically and analytically (solving for a Fourier expansion). This should also be generalized for 3D, for which the study of the new shapes of mode \( n = 1 \) promises many applications in describing droplet dynamics. In this spirit, it should also be possible to derive a scale-invariant flow-profile for the normal modes of a 3D drop in the linear regime, by generalizing our 2D derivation. It would also be of value to extend the study of these dynamics to compressible flows, as it was postulated that these might generate vortices.

**Conclusion**

This work introduced chain-oscillations, which were studied on a capillary jet, and their stability and traits were discussed. In a nutshell, chain-oscillations are the fundamental mode of vibration of a capillary jet. They are non-linear capillary oscillations, which are excited when the cross-section of the jet is stretched. As we have shown, there is a lot of physical richness surrounding the liquid chain which, although related, differs from the Rayleigh-Plateau instability and a liquid spiral.

A simple experiment allowed us to validate Niels Bohr’s non-linear corrections on the amplitude dependent frequency of chain-oscillations, seeing as they approximate the observed behaviour well. In this doing, we showed how the ellipse is not the natural shape to excite these vibrations, and suggest the usage of orifices which solve equation (2.22) (or generalizations thereof), which is valid for any mode \( n \). We also identified and tackled the issue of a diminished cross-section of a jet (compared to the area of the orifice), proposed a simplified explanation and suggest it to be solved through a singular flow solution. The different 1st wavelength of the chain-figure was attributed to a singularity at the nozzle-exit, meaning that Bohr was right in assuming
that the "presence of the orifice" introduces a small perturbation in the shape of the stationary chain. Here, experimental evidence of our model is still lacking, as well as a more rigorous theoretical derivation.

An alternative derivation of the normal modes of capillary liquids was presented (in particular for a 2D drop), which unveiled the scale-invariant nature of their flow-profile in the linear regime. This also motivated the identification of a maximum amplitude per mode (or maximum pressure or energy). The normal modes of a capillary drop were re-interpreted to correspond to the shapes of maximum compactness for a given $n$-fold symmetry, and which can correspond to a certain action or geometric effect (i.e. $n = 1$ for the movement about the center of mass and $n = 2$ for the stretching of a drop). In that regard, by studying the normal modes of a 2D drop, new shapes were found for the movement about the center of mass of a drop. This motivates that drops feel a net-force in a given direction, they deform "naturally" into "cusp-like"-, "drop-like"- and "heart-like" shapes (depending on the associated boundary conditions). This was argued to generalize to 3D and in fact many examples were given, in which such shapes do indeed manifest.

Additionally, we explored many examples of physical systems, where the normal modes of a capillary drop manifest (at least qualitatively). Most of those correspond to previously acknowledged polygonal shapes (or star-shapes). At small amplitudes, these are universal, scale-invariant polygonal shapes and they can be found to emerge throughout physics and nature! Examples of such shapes can be found for all known phases of matter and on length-scales from a few micrometers (for BEC’s) up to planetary scales. We have presented arguments for understanding the emergence of these shapes and suggested a few generalizations thereof. Note that these shapes are very often related to normal modes of different physical systems, and seem to unveil the presence of some form of resonance. As such, these modes seem to relate geometry to frequency and symmetry to compactness. Mostly, we believe that understanding the normal modes of a drop presents valuable insight and intuition for many such physical systems.

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A theoretical framework of the dynamics of these modes are studied in [1], by Lord Rayleigh (1879) and [2], by Niels Bohr (1909). Lord Rayleigh first described the appearance of normal modes in liquids for small amplitudes (in cylindrical and spherical coordinates) [1]. He introduced them, respectively, as the normal modes of a cylindrical jet and a spherical drop. Niels Bohr then built on his analysis using cylindrical- and polar coordinates, to ameliorate a method for the determination of surface tension [2]. In fact, Bohr studied chain-oscillations to do so, although he referred to them as “the case n = 2”.

Lord Rayleigh used a “potential flow Ansatz”, which is subject to the harmonic equation (for incompressible fluids):

\[
\begin{cases}
\vec{u} = \nabla \Phi \\
(\nabla \cdot \vec{u}) = 0
\end{cases} \implies \nabla^2 \Phi = 0 .
\]  
(A.1)

Here, \( \Phi \) is the flow potential and we will denote the velocity field as \( \vec{u} = \vec{u}(x, t) \) throughout this document. With this approach, Rayleigh determined the surface energy, the kinetic energy and the pressure distribution in both coordinate systems for small amplitudes. Using the Euler-Lagrange equations he then obtained the oscillation frequencies and showed how the Young-Laplace pressure drop is satisfied in this approximation. In cylindrical coordinates, and assuming that the modes have the shape \( R(\phi, z) = a + b_n \cos(n\phi) \cos(kz) \), for \( n \in \mathbb{N}_0 \), he obtained the potential energy due to the excess surface and the frequency of each mode:

\[
\begin{align*}
E_{0,surf} &= \gamma S_0 \simeq \frac{\pi \gamma}{2a} (k^2a^2 - 1) b_0^2 \delta z \\
E_n &= \gamma S_n \simeq \frac{\pi \gamma}{4a} (k^2a^2 + n^2 - 1) b_n^2 \delta z \\
\omega_n^2 &= \frac{\gamma}{\rho a^3} \frac{ika J'_n(ika)}{J_n(ika)} (k^2a^2 + n^2 - 1) \simeq \frac{\gamma}{\rho a^3} n(k^2a^2 + n^2 - 1) \simeq \frac{\gamma}{\rho a^3}(n^3 - n) .
\end{align*}
\]  
(A.2)

Note that the potential energy due to the excess surface \( E_{n,surf} \) is given per unit length \( \delta z \) along the cylindrical jet. Here, \( \gamma \) is the surface tension, \( \rho \) is the density of the liquid, \( \omega_n = 2\pi \xi_n \) is the angular frequency of the oscillation of mode \( n \), while \( S_n \) is the excess surface area produced by that respective mode. \( J_n(x) \) represents the Bessel-function (of the 1st kind and of order \( n \)), and \( J'_n \) its derivative. One can see how \( \omega_n \) becomes imaginary when \( n = 0 \) and \( |ka| < 1 \), which corresponds to the Rayleigh-Plateau Instability. This can also be seen from the potential energy \( E_{0,surf} \), which becomes negative under the same conditions, meaning that the system can lower its energy by exciting these unstable modes (\( n = 0 \)).

Similarly, the same reasoning was applied to a drop, in spherical coordinates. Assuming that the modes deform the boundary of a drop, from a sphere to \( R(\phi, \theta) = a + b_n P_n(\cos(\theta)) \), he obtained:

\[
\begin{align*}
E_{n,surf} &= \gamma S_n \simeq 2\pi \gamma \frac{(n-1)(n+2)}{2n+1} b_n^2 \\
\omega_n^2 &\simeq \frac{\gamma}{\rho a^3} n(n-1)(n+2) ,
\end{align*}
\]  
(A.3)

where \( P_n(x) \) represents the corresponding Legendre Polynomial of order \( n \). One can notice, how in both cases the movement about the center of mass \( (n = 1) \) is critically damped. The small amplitude solution suggests, that no excess surface needs to be created, in order to move the drop (cylinder) about its center of mass. In Rayleigh’s words:

“The case \( n = 1 \) corresponds to a displacement of the jet as a whole, without alteration in the form of the boundary. Accordingly, there is no potential energy and the frequency of vibration is 0.”

Thirty years later, Niels Bohr built on Lord Rayleigh’s work for his Master’s dissertation. His goal was to determine the dynamics of these oscillations with more mathematical rigor in cylindrical coordinates using the Navier-Stokes equation. Even though his work was motivated for the experimental measurement of the
surface tension of water, his findings improved the understanding of these modes quite substantially. Bohr showed that the density of the outer phase influences the frequency of oscillation, by promoting the density to the sum of the densities of the inner and outer phases. In the case of a liquid jet surrounded by air, we have:

\[
\rho_{\text{liquid}} \quad \rightarrow \quad \bar{\rho} = \rho_{\text{liquid}} + \rho_{\text{air}}
\]

\[
\omega_{n} \propto \sqrt{\frac{\gamma}{\rho a^{3}}} \quad \rightarrow \quad \sqrt{\frac{\gamma}{\rho a^{3}}}
\]  

(A.4)

In addition, Bohr studied the influence of viscosity on the oscillations of these modes (in the small amplitude regime), which is to damp the oscillations. He solved the liquid jet for a stationary solution (time independent), showing how the surface of the liquid jet is well described by the function:

\[
R(\phi, z) = a + b e^{-k_{z}z} \cos (k_{r}z) \cos (n\phi)
\]  

(A.5)

with:

\[
\begin{cases}
 k_{0} \simeq \frac{1}{v_{z}} \sqrt{\frac{\gamma(n^{3} - n)}{\rho a^{3}}} \sqrt{1 + \frac{(3n - 1)k_{0}^{2}a^{2}}{2n(n^{2} - 1)} + ...} \\
 k_{r} \simeq k_{0} \left[ 1 - \frac{n(n - 1)^{2}}{2} \left( \frac{2\eta}{\rho v_{z}a^{2}k_{0}} \right)^{2} n(n - 1)^{2} - \frac{2\eta}{\rho v_{z}a^{2}k_{0}} \right]^{2} + ... \\
 k_{i} \simeq \frac{2n(n - 1)}{\rho \frac{v_{z}a^{2}k_{0}}{2}} \left[ 1 + f_{n}(k_{0}a^{2}) \right] \left[ 1 - g_{n} \left( \frac{2\eta}{\rho v_{z}a^{2}k_{0}} \right) \right]
\end{cases}
\]  

(A.6)

where \(v_{z}\) represents the (average) velocity of the jet and \(\eta\) denotes the viscosity. Note that Bohr gives the 1st term(s) of the Taylor expansion of \(f_{n}\) and \(g_{n}\) for small arguments. Then, Bohr focused on the boundary shape of the cross-section of the jet. This corresponds to the shape of a “2D drop” in polar coordinates. Using perturbation theory to procure better approximations at higher amplitudes, he obtained:

\[
\begin{cases}
 R(\phi, t) = a - \frac{b^{2}}{8a} + b \cos (n\phi) \cos (\omega_{n}t) + \\
 + \frac{b^{2}}{a} \left[ a_{n} \cos (2n\phi) \cos (2\omega_{n}t) + b_{n} \cos (2n\phi) - \frac{1}{8} \cos (2\omega_{n}t) \right] + \mathcal{O}\left( \frac{b^{3}}{a^{2}} \right) \\
 \omega_{n}^{2} \simeq \frac{\gamma}{\rho \alpha^{3}} (n^{3} - n) \left[ 1 - \frac{b^{2}(n^{2} - 1)(34n^{2} - 33n^{2} + 50n - 18)}{16(2n^{2} + 1)(2n - 1)} + \mathcal{O}\left( \frac{b^{4}}{a^{4}} \right) \right]
\end{cases}
\]  

(A.7)

where he gives \(a_{n}\) and \(b_{n}\). This shows, that at finite amplitudes the oscillations stop being isochronous, in other words, the period of oscillation now depends on the amplitude of the oscillations (and so does the frequency). Additionally, the shape of the boundary of a given mode \(n\) is no longer sinusoidal. Instead, it is a more complicated periodical function, which can be expanded as a Fourier-sum, as seen in (A.7). Bohr concluded that the most general expression, combining all corrections for the frequency of oscillation will thus be a complicated expression of the form:

\[
\omega_{n}^{2} = \frac{\gamma}{\rho \alpha^{3}} (n^{3} - n) \left( 1 + \sum_{i=1}^{\infty} N_{i}(ka)^{2i} \right) \left( 1 - \sum_{j=1}^{\infty} M_{j} \frac{b^{2j}}{a^{2j}} \right) \left( 1 - L \left( \frac{2\eta}{\rho v_{z}a^{2}k_{0}} \right) \right)
\]  

(A.8)

where in general, \(N_{i}\), \(M_{j}\) and \(L\) are functions of \(n\), and \(M_{j}\) is also a function of \((ka)^{2}\).

Note: for readers interested in reading Bohr’s work [2], it should be noted that the Laplacian is represented by \(\nabla\).

**A.2 Derivation**

In this section, we will derive the mathematical description of the normal modes of an incompressible 2D drop. In doing so, we will first focus on the case \(n = 2\), which relates to chain-oscillations. In this derivation, we used yet a different approach from the previous references, in which we arrive at the same (small amplitude) solution by using a time-dependent Stokes Flow. In order to describe chain-oscillations (and these modes), we need the following system of equations:
\[
\begin{aligned}
\rho \frac{D\vec{u}}{Dt} &= \rho \left[ \partial_t + (\vec{u} \cdot \vec{\nabla}) \right] \vec{u} = \\
&= -\vec{\nabla}p + \eta \nabla^2 \vec{u} + \left( \zeta + \frac{\eta}{3} \right) \vec{\nabla} (\vec{\nabla} \cdot \vec{u}) + \vec{f}, \quad \text{Navier-Stokes (A.9)}
\end{aligned}
\]

where \( \rho \) represents the density, \( \eta \) the viscosity, and \( \zeta \) the bulk viscosity of the fluid, respectively. \( \vec{u} \) is the velocity field, and \( p \) the pressure. \( \sigma_{ij} \) denotes the stress tensor, and \( \vec{R} = \vec{R}(t, \vec{x}) \) describes the free surface, whose normal vector is \( \vec{n} \). Assuming incompressibility, we get from the first two equations:

\[
\begin{aligned}
\rho \left[ \partial_t + (\vec{u} \cdot \vec{\nabla}) \right] \vec{u} &= -\vec{\nabla}p + \eta \nabla^2 \vec{u} + \vec{f} \\
(\vec{\nabla} \cdot \vec{u}) &= 0 \quad \text{Continuity Equation (A.10)}
\end{aligned}
\]

Applying the divergence on the incompressible Navier-Stokes equation, we can combine it with the incompressibility condition, to get:

\[
\begin{aligned}
\nabla^2 \left( p + \frac{1}{2} \rho u^2 \right) + \rho \left[ \vec{u} \cdot (\vec{\nabla} \times \vec{\omega}) - \omega^2 \right] &= \left( \vec{\nabla} \cdot \vec{f} \right) \quad \text{(A.11)}
\end{aligned}
\]

where \( \vec{\omega} \equiv \vec{\nabla} \times \vec{u} \) is the vorticity, and where we used the following identities:

\[
\begin{aligned}
\vec{\nabla} \cdot (\vec{u} \times \vec{\omega}) &= \omega^2 - \vec{u} \cdot (\vec{\nabla} \times \vec{\omega}) \\
\frac{1}{2} \nabla^2(u^2) &= \vec{\nabla} \cdot [(\vec{u} \cdot \vec{\nabla}) \vec{u} + \vec{u} \times \vec{\omega}] \quad \text{(A.12)}
\end{aligned}
\]

To describe pure oscillations, we have to put \( \vec{\omega} = 0 \), obtaining in full generality, for an incompressible fluid:

\[
\nabla^2 \left( p + \frac{1}{2} \rho u^2 \right) = \left( \vec{\nabla} \cdot \vec{f} \right) \quad \text{(A.13)}
\]

Note now, that usually we have gravity acting as a bulk force \( \vec{f}_g = \rho \vec{g} \), but its divergence is 0. Thus, unless we apply a bulk force with non-zero divergence in the interior of our fluid, \( \nabla^2 (p + \frac{1}{2} \rho u^2) = 0 \) is always valid inside the fluid (in the bulk). This implies, that if we were to deform a drop (with forces acting on its surface) and maintain it in that deformed shape, since nothing is flowing:

\[
\vec{u} = 0 \quad \Rightarrow \quad \nabla^2 p = 0 \quad \text{(A.14)}
\]

This is very intuitive, as the Laplacian equation describes equilibrium dynamics very naturally. Notwithstanding, since the term \( \nabla^2 (\frac{1}{2} \rho u^2) \) comes from the convective derivative, we may ignore it for small velocities altogether (or small Reynolds number). In other words, in a non-turbulent regime these oscillations are described by a pressure that obeys the harmonic equation (or Laplacian equation):

\[
\nabla^2 p \approx (\vec{\nabla} \cdot \vec{f}) = 0 \quad \text{(A.15)}
\]

In essence, this means that at low energy perturbations the pressure distribution is still in (dynamic) equilibrium. Perturbations are thus felt instantly everywhere in the bulk, as convection processes are negligible. Note how up to now the dimension and geometry this problem is to be solved in, were not specified. Ergo, until this point, the equations are the same for 2D & 3D and for any coordinate system. Neglecting the convective derivative in the Navier-Stokes equation can also be justified by the Reynolds number. In our experiments high flow rates were used (large Reynolds number), still, by estimating the Reynolds number of the flow of a typical chain-oscillation (as in our experiments), we get:
\[
\begin{aligned}
\rho &\approx 10^3 \text{ kg/m}^3 \\
\eta &\approx 10^{-3} \text{ Pa s} \\
L &= R_0 \sim \text{cm} = 10^{-2} \text{ m} \\
\omega &\sim 10 \text{ Hz} \\
V &= \omega R \sim 1 \text{ dm/s} = 10^{-1} \text{ m/s}
\end{aligned}
\] 
\Rightarrow \quad Re = \frac{\rho VL}{\eta} = \frac{\rho \omega R_0^2}{\eta} \leq 10^3 \quad . \quad (A.16)

Noting that \( Re \leq 2000 \) is generally considered the definition of non-turbulent flows, we arrive at the exact same conclusion. In other words, turbulent flows are known to dominate at Reynolds numbers \( Re \geq 2000 \), and therefore, we can ignore the convective derivative in Navier-Stokes, seeking solutions for the time-dependent Stokes equation. Fortunately, as we just saw, this means that chain-oscillations are described by a harmonic pressure profile.

\[
\begin{aligned}
\rho \partial_t \vec{u} &= -\vec{\nabla} p + \eta \nabla^2 \vec{u} \\
\vec{\nabla} \cdot \vec{u} &= 0
\end{aligned}
\] 
\Rightarrow \quad \nabla^2 p(\vec{r}, t) = 0 \quad . \quad (A.17)

Note that the partial time derivative is kept, as there is an additional source of temporal dependence: surface tension, at the boundary interface between the water and the surrounding air introduces both a restoring force (for shapes deviating from the cylinder) and an according pressure profile in the bulk of the fluid phases (mostly inside the liquid, the inner phase, but also in the surrounding air, or outer phase). Chain-oscillations are thus described by a harmonic pressure profile, because it obeys the harmonic Equation (or Laplacian Equation). There is a general solution (expansion) for this differential equation, which in polar coordinates takes the form:

\[
\nabla^2 p = 0 \implies p(r, \phi) = (\alpha \phi + \beta) \log(r) + \sum_{n \in \mathbb{Z}} r^n \left[ A_n \cos(n\phi) + B_n \sin(n\phi) \right] \quad . \quad (A.18)
\]

- The \( \log(r) \) term is known to describe sinks/sources \( (\beta \neq 0) \) and vortices \( (\alpha \neq 0) \), or combinations of both.
- \( n = 0 \) describes constant pressure and is therefore the equilibrium shape, the circle.
- \( n = 1 \) is the only term that can describe movement about the center of mass, since it is the only mode able to satisfy \( \vec{\nabla} p = \text{constant} \neq 0 \) everywhere.
- \( n \geq 2 \) describe symmetric deformations, that can produce "star-shaped" oscillations and polygonal patterns.
- \( n \leq -1 \) describe the outer phase of their positive counterpart. They become important for hollow polygonal patterns (or "star-shapes").

Let it be noted that all terms in the sum over \( n \), if present, produce movement in both the radial and angular directions. Additionally, there are 2 ways to have either only radial flow, or only tangent flow (in the direction of \( \phi \)), which correspond to the logarithmic contributions. Combinations of these modes are also allowed. Although all of these terms solve for a harmonic flow in 2D, the only modes relevant for a drop are the ones with \( n \geq 0 \). Let us now proceed to deriving the linear solution of 2D chain-oscillations in time. Starting in a cylindrical coordinate system:
If the velocity in the \( z \)-direction is taken as constant and \(|ka| \ll 1\), the motion is essentially decoupled. That is, the phenomenon can be thought of as a constant flow in \( \vec{e}_z \) and an additional 2D oscillation of the cross-section (in polar coordinates). Note that \(|ka| = \frac{2\pi a}{\lambda} \ll 1\) is usually satisfied and it is equivalent to ignoring a term in the Young-Laplace pressure drop:

\[
\frac{2\pi a}{\lambda} \ll 1 \implies \kappa_r \gg \kappa_z \implies \kappa_{cyl.} \simeq \kappa_{polar}.
\]

The curvature in \( \vec{e}_z \), \( (\kappa_{r,z}) \), thus matters only for very small wavelengths \( (\lambda \sim a) \), and in that case it should contribute about equally, so that we can set \( \kappa_{cyl.} \simeq 2\kappa_{r,\phi} \). This is why we can focus on the 2D system, without too much loss of information. Noting that decoupling the problem to 2D polar coordinates amounts to assuming a constant jet velocity everywhere, in this approximation, chain-oscillations can be thought of as a laminar flow with super-imposed oscillations in its cross-section. Previously, we combined the Continuity Equation with Navier-Stokes, showing that the pressure profile inside an “ideal” 2D chain-oscillation is:

\[
p(r, \phi, t) = \frac{\gamma}{\alpha} + A_2(t) r^2 \cos (2\phi + \phi_0) \implies \vec{\nabla} p = 2A_2(t) r \left( \cos (2\phi) - \sin (2\phi) \right),
\]

in polar coordinates, and we can set \( \phi_0 = 0 \). Progressing, we would have to introduce the Boundary conditions for the free surface and the equation of motion for the free surface. In 2D these assume the form:

\[
\begin{aligned}
\sum n_i \sigma_{ij}^{in} &= \sum n_i \sigma_{ij}^{out}, \quad \text{Free surface condition} \\
\left[ \bar{n} \bar{R}(\phi, t) - \bar{u} \right] \cdot \bar{n} &= 0, \quad \text{Equation of motion for the free surface}
\end{aligned}
\]

\[
\implies \begin{cases}
[\Delta p - \sigma_\eta] = \gamma \kappa \bigg|_{r=R(\phi, t)} \\
\frac{\partial_t}{R(\phi, t)} \begin{pmatrix} R(\phi, t) & 0 \\ 0 & u_r \end{pmatrix} - \begin{pmatrix} 0 \\ u_{\phi} \end{pmatrix} \cdot \frac{1}{R(\phi, t)} \frac{\partial_p R(\phi, t)}{R(\phi, t)} = 0
\end{cases}
\]

where \( \sigma_\eta \) represents viscous stresses. Ignoring the surrounding outer fluid phase (air):

\[
\implies \begin{cases}
[p - \sigma_\eta - \gamma \kappa]_{r=R} = 0 \\
\frac{\partial_t}{R} + u_\phi \frac{\partial_p \log (R)}{R} = u_r
\end{cases}
\]

As this is an illustrative example, we will take further approximations, and ignore the contribution of viscosity. We also ignore the term \( u_\phi \frac{\partial_p \log (R)}{R} \), which is small. Note how this relates to the condition of finite curvature in equation (2.28). This leaves us with:
\[ \frac{1}{\gamma} p[R(\phi)] \simeq \kappa[R(\phi)] \simeq \frac{a - \delta R(\phi) - \tilde{\delta} R(\phi)}{a^2}, \forall_t \]  
\[ \partial_t R \simeq u_r, \]  
\[ \Rightarrow \begin{cases} 
A_2(t) \frac{R^2 \cos(2\phi)}{\gamma} \simeq \left( \kappa[R(\phi)] - \frac{1}{\gamma} \right) \simeq -\frac{\delta R(\phi) + \tilde{\delta} R(\phi)}{a^2} 
2A_2(t) r \left( \frac{\cos(2\phi)}{-\sin(2\phi)} \right) + \rho \partial_t \bar{u} = 0 \end{cases} \]  
\[ \Rightarrow \begin{cases} 
\frac{A_2}{\gamma} a^2 = \frac{3b}{a^2} 
2A_2 a = \rho \omega^2 b \end{cases} \]  
\[ \Rightarrow \begin{cases} 
A_2 = \frac{3b\gamma}{a^2} 
\omega^2 = \frac{6\gamma}{\rho a^3} \end{cases} \]  
This leaves us with the self-consistent solution to the problem of 2D chain-oscillations (up to linear order):

\[ \begin{cases} 
R(\phi, t) = a - \frac{b^2}{8a} + b \cos(\omega t) \cos(2\phi) 
u_r(r, \phi, t) = -b \omega \frac{r}{a} \sin(\omega t) \cos(2\phi) 
\nu_\phi(r, \phi, t) = b \omega \frac{r}{a} \sin(\omega t) \sin(2\phi) 
p(r, \phi, t) = \frac{\gamma}{a} + \frac{3\gamma b}{a^2} \left( \frac{r}{a} \right)^2 \cos(\omega t) \cos(2\phi) \end{cases} \]  
\[ \begin{cases} 
A_2 = \frac{3 \gamma b}{a^3} 
\omega = 2\pi f_2 = \sqrt{\frac{6\gamma}{\rho a^3}} \left( = \omega_2 \right) \end{cases} \]  
Note that both the pressure and the flow profile (velocity field) are scale invariant. This also holds for the general case of a 2D oscillation of a mode \( n \), for which the solution is:
\[
R(\phi, t) = a - \frac{b^2}{8a} + b \cos(\omega_n t) \cos(n\phi)
\]
\[
u_r(r, \phi, t) = -b \omega_n \left(\frac{r}{a}\right)^{n-1} \sin(\omega_n t) \cos(n\phi)
\]
\[
u_\phi(r, \phi, t) = b \omega_n \left(\frac{r}{a}\right)^{n-1} \sin(\omega_n t) \sin(n\phi)
\]
\[
p(r, \phi, t) = \frac{\gamma}{a} + \frac{(n^2 - 1) \gamma}{a^2} b \left(\frac{r}{a}\right)^n \cos(\omega_n t) \cos(n\phi)
\]
\[
A_n = \frac{(n^2 - 1) \gamma b}{a^{n+1}}
\]
\[
\omega_n = 2\pi f_n = \sqrt{\frac{n(n^2 - 1)\gamma}{\rho a^3}}
\]

(A.31)
Appendix B: Materials and Methods

B.1 Experimental setup

To produce chain-oscillations in the Laboratory, we had our workshop engineer a jet-stabilizer (also known as a Laminar flow nozzle, or simply laminar nozzle). This piece consists of a hollow transparent tube (out of plexiglass), which is filled with narrow metal grids, sponges and tightly packed straws. While the liquid is forced through this device, the flow is smoothly evened out and becomes effectively laminar at the exit. In other words, the narrow spacing imposes a much smaller length-scale for the flow, effectively reducing the Reynolds number drastically and thus suppressing the formation of turbulence.

As can be seen from the picture, the ends of this device were adapted, so that a hose can be connected at the entry, and a thin plate (with the desired orifice shape) at the exit. The orifices were cut with a precision wire cutter out of a thin stainless steel plate, producing the desired elliptical shapes up to a precision of $1\mu m$. In order to produce the desired chain-oscillations, this device was clamped vertically over a water tank and connected to a tap-water source, as depicted in Figure B.2.
Additionally, a flow-meter was used to measure the flow-rate, while the remaining quantities were measured through pictures taken with a high-resolution camera. More details on the measurements can be found in the following section.

B.2 Measurements

The physical quantities measured here are the flow-rate $Q$, the wavelength $\lambda$, and $R_{\text{max}}$ & $R_{\text{min}}$ ($R_{\text{max/min}}$ are respectively the maximum and minimum radial dimensions of a chain-oscillation over one wavelength). The flow-rate was measured with a flow-meter (Krohne H250/RR/M40/ESK), and the last 3 quantities were obtained from taking pictures next to a scale. All other quantities can either be obtained from these, or taken as constants (for instance the density $\rho$ and the surface tension $\gamma$). For the physical measurements, we thus varied the inflow of water, then waited briefly until the shape of the chain-oscillations and the measured flow-rate were stabilized. The stabilization was noticed to take up to about 5 seconds (and an average of about 1 second). This should correspond to the time necessary for the pressure-distribution to even out along the setup and reaching its stationary solution, starting from the time the flow-rate was changed. Naturally, this adjustment period (in which unstable contributions to the pressure damp out) was proportional to the change in flow-rate, $\Delta Q$.

Looking at the flow-meter and the stationary shape of the chain-figure, a picture was taken and the flow-rate recorded (associated to that picture). For this to work, however, the picture has to be taken at 0° (or 90°) to the oscillatory motion (using the definition in Figure 1.3). Be this not the case, the formula describing the surface of the jet $R(\phi, z)$ is no longer the one presented in the following section. Although this does not affect the measurement of the wavelength, it does have consequences for $R_{\text{max/min}}$: if the chain-figure is not photographed at 0° (or 90°), then the produced picture does not display $R_{\text{max}}$, but its (smaller) rotated projection. Similarly, instead of being the minimum thickness of the chain-figure, $R_{\text{min}}$ will seem larger than it should. This is due to the geometry of a chain, which has links at 90° to each other: $R_{\text{min}}$ at 0° corresponds to $R_{\text{max}}$ at 90° (and vice-versa), as can be observed in Figure 1.3 and Figure 1.4. At 45° the projections of $R_{\text{max}}$ and $R_{\text{min}}$ are equal, which is why the chain-figure appears to be symmetric from that perspective. Therefore, the camera was set up with a mount and calibrated to be as even as possible and at 90° to the elongations of the liquid chain-figure.

B.3 Calculations for the experiment

When we discussed our experiment in Part (I), some of the underlying mathematical details were left out or oversimplified for clarity and simplicity. In this section we discuss those details and the computations surrounding them.

B.3.1 Additional effects on the frequency

As claimed in Part (I), Lord Rayleigh’s linear estimate (1.3) represents the simplest theoretical prediction of the frequency, which can then be corrected to account for additional effects, such as the surrounding air, gravity, small wavelengths, the viscosity, and large amplitudes. This presents quite a large variety of different theoretical estimates. However, we will now show that in the case of our experiments most of these corrections are insignificant and can be ignored.

The surrounding air

For the case of a water-jet surrounded by air, we find that the correction (A.4) due to the surrounding outer fluid phase (the air) is insignificant. That is, for our experiments the correction in density can be ignored, seeing as $\rho_{\text{water}} + \rho_{\text{air}} \simeq \rho_{\text{water}}$. Another way of putting it, is that $\sqrt{\frac{\rho_{\text{air}}}{\rho_{\text{water}}}} \simeq 0.9994$, implying that the correction on the frequency of oscillation due to the surrounding air only amounts to about 0.06%.

Gravity

It is known that a jet will experience thinning due to the effect of gravity. As the jet thins, its cross-section will diminish, implying a smaller radial scale $a$, and thus slightly increasing the frequency. However, by measuring the radial scale $a$ in latter wavelengths than the one over which the measurements were taken, we found that there was no significant change. Given that $Q = v_z A$ and that the chain-figure is stationary, we can assume
the velocity \( v_z \) to be constant over one wavelength. This implies that there is no significant gravitational acceleration over one wavelength. Therefore, we found that in practice the effect of gravitational pull was negligible for our experimental conditions.

**Viscosity**

Let us inspect the influence of viscosity \( \eta \) on the frequency of chain-oscillations and determine whether it should be taken into account in our experiments. Niels Bohr’s calculations [2] show that the influence of viscosity is to damp the oscillations. As such, there is a correction in frequency due to the damping effect of viscosity, which is expected to lower the frequency. As seen in (A.6), the correction should be small when:

\[
\frac{n(n-1)^2}{2} \left( \frac{2\eta}{\rho v_z a^2 k_0} \right)^2 \simeq \left( \frac{\eta}{\pi \rho a^2 f} \right)^2 \ll 1 ,
\]

where we introduced \( n = 2 \) and used \( v_z k_0 \simeq \omega = 2\pi f \). However, there is another way through which viscosity could influence the frequency of chain-oscillations. Since \( f_{\text{Bohr}} \) depends on the amplitude, viscous damping could also affect Bohr’s corrected frequency estimate by increasing the frequency. Since our measurements are taken over one wavelength \( \lambda \), we also need to require that the change in amplitude over \( \lambda \) is negligible. Looking at (A.6) again, this corresponds to the condition:

\[
k_i \lambda \ll 1 = \Rightarrow \eta \frac{2n(n-1)}{\rho} v_z a^2 \lambda \ll 1 .
\]

Introducing \( n = 2 \) and using \( v_z \simeq \lambda f \), we obtain a very similar condition to (B.1):

\[
\frac{4\eta}{\rho a^2 f} \ll 1 .
\]

As a matter of fact, if (B.3) is satisfied, so is (B.1). We now proceed with the condition (B.3), and setting \( f \sim f_{\text{Rayleigh}} = \frac{1}{2\pi} \sqrt{\frac{\rho \gamma a}{\rho^3}} \), this condition implies:

\[
\frac{32\pi^2 \eta^2}{3 \rho \gamma a} \ll 1 .
\]

Introducing the values:

\[
\begin{align*}
\rho & \sim 10^3 \text{ kg/m}^3 \\
\eta & \sim 10^{-3} \text{ Pas} \\
a & \sim 10^{-3} \text{ m} \\
\gamma & \sim 10^{-1} \text{ N/m} \\
\frac{32\pi^2}{3} & \sim 10^2 \text{ Hz}
\end{align*}
\]

we obtain from (B.4):

\[
\frac{32\pi^2 \eta^2}{3 \rho \gamma a} \sim 10^{-3} \ll 1 .
\]

Consequently, the viscous corrections on the frequency of chain-oscillations can be assumed negligible, which is why we ignore them in our experimental discussion in Part (I).

**Small wavelengths**

Here we discuss the effect of small wavelengths compared to the jet’s radius. Such small wavelengths imply that the curvature in \( z \) becomes important, which yields a stronger restoring pull and thus a higher frequency. This is seen in the theoretical description to correspond to the importance of the adimensional number \( |ka| = \left| \frac{2\pi a}{\lambda} \right| \). Bohr’s calculations show that when \( |ka| \) cannot be ignored, the frequency suffers corrections, which are expressed as an infinite expansion in \( (ka)^2 \). The largest correcting term of that expansion is presented in equation (A.2), and assumes the from:
\[
\omega_n^2 \simeq \frac{\gamma}{\rho a^3} n \left( k^2 a^2 + n^2 - 1 \right) .
\]  

(B.7)

The effect of the correction in (B.7) due to \(|ka|\) is due to the curvature in \(z\). Therefore, it should become important at low flow-rates and small amplitudes, otherwise the radial curvature is expected to dominate. Even though we found this effect to be of low importance for our experimental data-set, there are a few data-points for which this correction is needed to explain the discrepancy in the measured frequency. These correspond to data-points for which the experimentally measured frequency \(f_a\) is larger than the theoretical predictions. This is only the case for the first few data-points (low flow-rates) in orifices 8, R15/7.5 and R10/7.5, as can be confirmed in the section Measured Data in this Appendix. To account for these discrepancies, the theoretical predictions should be adjusted in the following way:

\[
\begin{align*}
\tilde{f}_{\text{Rayleigh}} &= \frac{1}{2\pi} \sqrt{\frac{2(k^2 a^2 + 3)\gamma}{\rho a^3}} , \\
\tilde{f}_{\text{Bohr}} &= \frac{1}{2\pi} \sqrt{\frac{2(k^2 a^2 + 3)\gamma}{\rho a^3}} \sqrt{1 - \frac{37 b^2}{24 a^2}} ,
\end{align*}
\]  

(B.8)

Note that in this way, the measurement of the wavelength \(\lambda\) is now also required for the theoretical predictions of the frequency. These corrections should account for the slightly higher frequency at low flow-rates. As such, the most accurate theoretical prediction for the frequency of chain-oscillations corresponds to the corrected form of \(\tilde{f}_{\text{Bohr}}\) in (B.8). However, since this correction only produced noticeable changes in 3 of our orifices (orifices 8, R15/7.5 and R10/7.5), and only for a few of the first data-points, it was left out of the discussion of our experiments. In contrast, Bohr’s correction due to large amplitudes produces quite significant changes and for the entirety of our data-set.

Hence, although chain-oscillations are a 3D phenomenon, we find that the curvature in the radial plane dominates considerably over the curvature along the jet axis. At high flow-rates this is to be expected, however, we find that even at low flow-rates the contribution of the curvature along the jet axis produces only very mild corrections. Given the orifice and the properties of the liquid, the wavelength is set by the flow-rate:

\[
\begin{align*}
\lambda &= \frac{v_z}{f} = \frac{Q}{\pi a^2 f} \propto \frac{Q}{a^3} , \\
ka &= \frac{2\pi a}{\lambda} \implies ka \propto \frac{a^3}{Q} ,
\end{align*}
\]  

(B.9)

where we used the fact that \(f \propto a^{-\frac{3}{2}}\). If \(a\) was constant, small wavelengths would be expected to manifest for low flow-rates. But our experiments show that \(a = a(Q) \to 0\) as \(Q \to 0\). And although the analytical form of \(a(Q)\) is still unknown, it can be Taylor-expanded around small flow-rates. Therefore, given that the expansion of \(a(Q)\) has to be at least linear in \(Q\) as \(Q \to 0\), we should expect \(ka \to 0\) in the same limit. To find the relationship \(\lambda(Q)\) we would first need to know \(a(Q)\). Consequently, the theory cannot guarantee at this point that the effect of \(|ka|\) is always small. For now, we can only note that in our experiment small wavelengths seem to be very difficult to excite. Still, it seems reasonable that long wavelengths are excited naturally, because small wavelength excitations have a greater energetic cost. Hence, our results suggest that chain-oscillations (in water) manifest close a laminar flow regime, and for small \(|ka|\).

Disregarding the few data-points in which the correction due to \(|ka|\) in (B.8) does have a slight influence, the corrections in frequency due to the surrounding air, gravity, viscosity and small wavelengths are seen to be insignificant. Therefore, we conclude that for chain-oscillations in water, and the orifices and flow-rates we investigated, the only significant correction is due to large amplitudes \(\varepsilon\).

### B.3.2 Calculations regarding the measurements

In the experimental section of Part (I) we presented an oversimplified model for the measurements of the parameters \(a\) and \(b\). Here, we will discuss the details of the actual model and the computations surrounding it. Lord Rayleigh’s solution, valid up to linear order in amplitude, is given by:

\[
\begin{align*}
R(\phi, z)_{\text{Rayleigh}} &= a - \frac{b^2}{8a} + b \cos (2\phi) \cos (kz) , \\
\tilde{f}_{\text{Rayleigh}} &= \frac{1}{2\pi} \sqrt{\frac{6\gamma}{\rho a^3}} ,
\end{align*}
\]  

(B.10)
in cylindrical coordinates, and valid for $|ka| \ll 1$ and in a laminar-flow regime, where $v_z = \lambda f$. The parameters $a$ and $b$ describing the liquid chain can be determined by taking a picture of the stationary chain-figure and measuring the largest (and smallest) radius that the chain-figure displays. For Rayleigh’s solution this implies:

\[
\begin{align*}
\frac{1}{2} (R_{\text{max}} - R_{\text{min}}) &= b \\
\frac{1}{2} (R_{\text{max}} + R_{\text{min}}) &= a (1 - \frac{1}{8} \frac{b^2}{a^2}) \quad .
\end{align*}
\]  

(B.11)

Bohr developed this approach up to 2nd order in amplitude, which yields an correction on the frequency, as well as additional 2nd order terms in the boundary shape $R(\phi, z)$:

\[
\begin{align*}
R(\phi, z)_{\text{Bohr}} &= R(\phi, z)_{\text{Rayleigh}} + \frac{b^2}{a} \frac{1}{6} \cos(4\phi) \cos(2kz) + \frac{1}{4} \cos(4\phi) - \frac{1}{8} \cos(2kz) + \mathcal{O}(\frac{b^3}{a^2}) \\
\frac{f_{\text{Bohr}}}{2\pi} &= \frac{6\gamma}{\rhoa^3} \sqrt{1 - \frac{37b^2}{24a^2}} \quad .
\end{align*}
\]  

(B.12)

The given form of $f_{\text{Bohr}}$ is obtained through introducing $n = 2$ in the generalized form of Bohr’s approximation:

\[
\begin{align*}
\omega_n^2 &\approx \frac{\gamma(n^3 - n)}{\rhoa^3} [1 - \frac{b^2}{a^2} (n^2 - 1)(34n^3 - 33n^2 + 50n - 18)] \\
&\quad / 16(2n^2 + 1)(2n - 1) \\
\Rightarrow f_{\text{Bohr}} &= \frac{1}{2\pi} \sqrt{\frac{6\gamma}{\rhoa^3}} \sqrt{1 - \frac{37b^2}{24a^2}} \quad .
\end{align*}
\]  

(B.13)

The parameters $a$ and $b$ can again be determined directly from the chain-figure, which for Bohr’s solution means:

\[
\begin{align*}
\frac{1}{2} (R_{\text{max}} - R_{\text{min}}) &= b \\
\frac{1}{2} (R_{\text{max}} + R_{\text{min}}) &= a (1 + \frac{1}{6} \frac{b^2}{a^2}) \quad .
\end{align*}
\]  

(B.14)

Measuring $R_{\text{max}}$ and $R_{\text{min}}$ off the cylindrical shape captured in the pictures, we can relate them to $a$ and $b$:

\[
\begin{align*}
& b = \frac{1}{2} (R_{\text{max}} - R_{\text{min}}) \\
& a = \frac{1}{4} \left( \Sigma + \sqrt{\Sigma^2 - \frac{2}{3} b^2} \right) \\
& \Sigma = R_{\text{max}} + R_{\text{min}} \quad .
\end{align*}
\]  

(B.15)

It is of importance to note that $a$ and $b$ have to be derived from a local measurement of $R_{\text{max/min}}$ inside the measured wavelength $\lambda$. This is crucial, as the frequency depends on the amplitude, which is not precisely the same for every consecutive wavelength (since the amplitude is damped continuously through viscous forces over time). As a consequence, traditional methods of averaging over consecutive wavelengths are not applicable. Notice further, that we measure the flow rate at the orifice exit, $Q$, and that the chain-figure is stationary and has a constant cross-section (to the degree of accuracy of our measurements). Its cross section is $A = \pi a^2$ = constant, when defined through (B.10) or (B.12). The errors were estimated as 2, 5% of the measured distances (on the pictures), that is, for $\lambda$, $R_{\text{max}}$ and $R_{\text{min}}$. The uncertainty in flow rate was the uncertainty of our flow-meter, $e_Q = 0, 005 \frac{1}{\text{min}}$. Then, these errors were propagated accordingly, through the maximum possible error estimate, given the formulas used:

\[
\forall f(x) : \quad e_f|_{\vec{x}=\vec{x}_0} := \sum_i \left| \frac{\partial f}{\partial x_i} \cdot e_{x_i} \right|_{\vec{x}=\vec{x}_0} \quad ,
\]  

(B.16)

remembering that if $f(x) \propto x^\alpha$, then:

\[
\frac{\partial f}{\partial x_i} \cdot e_{x_i} \big|_{\vec{x}=\vec{x}_0} = f \alpha e_{x_i} \frac{\vec{e}_{x_i}}{\vec{x}_i} \quad .
\]  

(B.17)
This lead to the error bars visible in the graphs depicting our data.

### B.4 Measured data

In total, 13 different elliptical orifices were used in these studies. The specifications of each orifice can be found in the table of Figure B.3. We fabricated 2 different sets of orifices. For the orifices named 1-9 the major axes ($D_x$) and the eccentricities were fixed at “round” values. Note that orifice 3 was left out, as the wire we used in the wire-cutter was too thick to produce it. The remaining orifices were set with round values for the major axes ($D_x$) and the minor axes ($D_y$), producing different eccentricities as a result.

<table>
<thead>
<tr>
<th>Name</th>
<th>$D_x$ (cm)</th>
<th>$D_y$ (cm)</th>
<th>eccentricity</th>
<th>$R_0$ (cm)</th>
<th>A (cm$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>R 15/5</td>
<td>3</td>
<td>1</td>
<td>0.943</td>
<td>0.866</td>
<td>2.36</td>
</tr>
<tr>
<td>R 15/7.5</td>
<td>3</td>
<td>1.5</td>
<td>0.866</td>
<td>1.061</td>
<td>3.53</td>
</tr>
<tr>
<td>R 15/9.5</td>
<td>3</td>
<td>1.9</td>
<td>0.774</td>
<td>1.194</td>
<td>4.48</td>
</tr>
<tr>
<td>R 10/7.5</td>
<td>2</td>
<td>1.5</td>
<td>0.661</td>
<td>0.813</td>
<td>2.08</td>
</tr>
<tr>
<td>R 5/7.5</td>
<td>1</td>
<td>1.5</td>
<td>0.745</td>
<td>0.407</td>
<td>0.52</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.436</td>
<td>0.9</td>
<td>0.330</td>
<td>0.342</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.312</td>
<td>0.95</td>
<td>0.279</td>
<td>0.25</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>0.872</td>
<td>0.9</td>
<td>0.660</td>
<td>1.37</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>0.624</td>
<td>0.95</td>
<td>0.981</td>
<td>0.56</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>0.282</td>
<td>0.99</td>
<td>0.376</td>
<td>0.44</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>1.744</td>
<td>0.9</td>
<td>1.320</td>
<td>5.48</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>1.248</td>
<td>0.95</td>
<td>1.118</td>
<td>3.92</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>0.564</td>
<td>0.99</td>
<td>0.751</td>
<td>1.77</td>
</tr>
</tbody>
</table>

Figure B.3: Table with the specifications regarding the elliptical orifices used in our experiments

Note that the given values of $R_0$ correspond to the effective length-scale of the orifices. These are defined such that $A_{ellipse} = A_{orifice} = \pi R_0^2$. In the following pages you may find the results of the measurements of all these orifices. For each of them, we display the following 3 graphs (in the following order from left to right):

- A plot of the radial dimensions of the chain-figure, $a(Q)$ and $b(Q)$.
- A plot of the chain-oscillations’ amplitude, $\varepsilon(Q) \equiv \frac{b(Q)}{a(Q)}$.
- A plot comparing the 3 frequencies, $f_{\lambda,b,r}(Q)$.

Naturally, all of the quantities in these graphs are presented as a function of the flow-rate, $Q$. The respective first graphs of each orifice show clearly that both parameters of the liquid chain-figure depend on the flow-rate. One can see that for all of these, the radial length-scale of the liquid chain $a$ does not equal the length-scale of the orifice $R_0$, when compared to the table in Figure B.3. This is fairly trivial, given that $R_0$ is constant, while $a = a(Q)$ depends on the flow-rate. In addition, one can see the irregular nature of both $a$ and $b$, which is addressed in the section that discusses the results of our experiment in Part (I).

It is plain to see that both $a$ and the amplitude of the chain-oscillations $b$ increase with increasing flow-rate. However, the amplitude $b$ is seen to increase faster in comparison to $a$. To see this, we show the respective second graphs, which plot the adimensional amplitude of the chain-figure as a function of $Q$. Given that $\varepsilon(Q) \equiv \frac{b(Q)}{a(Q)}$ also increases with increasing flow-rate, the amplitude $b$ is seen to increase faster in comparison to $a$.

In the case of the third graphs, we are comparing the 3 different frequency estimates for each orifice. The error bars for the blue and orange lines were not included for clarity. The grey line was the one chosen to
display error bars in the $f$-axis, since its uncertainties are always greater than the other 2 lines, and because it is the main subject of study: Niels Bohr’s corrected estimate for the frequency of chain-oscillations, $f_{\text{Bohr}}$. The blue lines represent the experimental measurement $f_{\lambda}$, by knowing the average flow-rate and the wavelength. The orange lines depict Rayleigh’s estimate, $f_{\text{Rayleigh}}$.

**B.4.1  R 15/5**

![Graph](image)

Figure B.4: $a(Q) \& b(Q)$, $\varepsilon(Q)$ and $f_{\lambda,\text{Bohr, Rayleigh}}$ respectively, for the orifice: **R15/5**

**B.4.2  R 15/7.5**

![Graph](image)

Figure B.5: $a(Q) \& b(Q)$, $\varepsilon(Q)$ and $f_{\lambda,\text{Bohr, Rayleigh}}$ respectively, for the orifice: **R15/7.5**

**B.4.3  R 15/9.5**

![Graph](image)

Figure B.6: $a(Q) \& b(Q)$, $\varepsilon(Q)$ and $f_{\lambda,\text{Bohr, Rayleigh}}$ respectively, for the orifice: **R15/9.5**

**B.4.4  R 10/7.5**

![Graph](image)

Figure B.7: $a(Q) \& b(Q)$, $\varepsilon(Q)$ and $f_{\lambda,\text{Bohr, Rayleigh}}$ respectively, for the orifice: **R10/7.5**
Figure B.8: $a(Q) \& b(Q), \varepsilon(Q)$ and $f_{\lambda, \text{Bohr, Rayleigh}}$ respectively, for the orifice: $R5/7.5$

Figure B.9: $a(Q) \& b(Q), \varepsilon(Q)$ and $f_{\lambda, \text{Bohr, Rayleigh}}$ respectively, for the orifice: 1

Figure B.10: $a(Q) \& b(Q), \varepsilon(Q)$ and $f_{\lambda, \text{Bohr, Rayleigh}}$ respectively, for the orifice: 2

Figure B.11: $a(Q) \& b(Q), \varepsilon(Q)$ and $f_{\lambda, \text{Bohr, Rayleigh}}$ respectively, for the orifice: 4
Figure B.12: $a(Q) \& b(Q)$, $\varepsilon(Q)$ and $f_\lambda$, Bohr, Rayleigh respectively, for the orifice: 5

Figure B.13: $a(Q) \& b(Q)$, $\varepsilon(Q)$ and $f_\lambda$, Bohr, Rayleigh respectively, for the orifice: 6

Figure B.14: $a(Q) \& b(Q)$, $\varepsilon(Q)$ and $f_\lambda$, Bohr, Rayleigh respectively, for the orifice: 7

Figure B.15: $a(Q) \& b(Q)$, $\varepsilon(Q)$ and $f_\lambda$, Bohr, Rayleigh respectively, for the orifice: 8
Figure B.16: $a(Q)$ & $b(Q)$, $\varepsilon(Q)$ and $f_\lambda$, Bohr, Rayleigh respectively, for the orifice.
Appendix C: Harmonic mode- and Laplacian eigenfunction-expansions

$∇^2 f(\vec{r}, t) = 0$ is a central equation across most domains of physics. Appendix C is dedicated to the understanding of the underlying mathematics surrounding the harmonic expansions, but also the Laplace Equation and its generalizations, the Helmholtz- and the Poisson Equation. We shall compare the general solutions of these equations in 2D and 3D coordinate-systems. In this way, we will be able to understand the relationship of similar polygonal shapes, geometry and dynamics across fields of physics.

C.1 Harmonic mode expansion

It is very well known, that we can solve the Harmonic equation (or Laplacian equation) in any dimension and coordinate system, with analytical expressions. The Harmonic equation is a homogeneous linear differential equation. This means that, if $f_h(\vec{r}, t)$ satisfies $∇^2 f(\vec{r}, t) = 0$, then a general and unique linear combination of an orthogonal (orthonormal) basis of functions exists, which equals the solution $f_h(\vec{r}, t)$. Conversely, this is to be equivalent to saying: Be $f_h(\vec{r}, t)$ a mathematical function, which satisfies

\[
\begin{align*}
∇^2 f_h(\vec{r}, t) &= 0 \\
\land \text{ the given boundary conditions in domain } Ω
\end{align*}
\] (C.1)

then there exists one unique linear superposition of the harmonic basis expansion, which converges to (equals) $f_h(\vec{r}, t)$:

\[
\implies f_h(\vec{r}, t) = \sum_{m,n}^{+∞} \alpha_{nm} g_{nm}(\vec{r}, t)
\] (C.2)

Note, that the precise form of the infinite sum, and of $g_{nm}$ depend on the choice of coordinate-system, which defines natural transformations. In other words, these equations and their results are independent of the coordinates we might choose to describe them in, because the Laplacian Equation is diffeomorphism invariant. Even though this is a requirement for most physical theories, it is an important fact to be acknowledged. It also implies, that the sets of harmonic expansions of different coordinate systems are equivalent bases, spanning the same function space. The choice of coordinates is really only about exploiting the symmetry of the problem, which simplifies its description, ergo, a way to describe a problem in its “natural” coordinates.

Note further, how the harmonic equation can be solved on the complex plane (in 2D). As a matter of fact, it is a very natural algebra, to solve these problems in. We are allowed to use special properties of analytic functions, originating from the symmetries of the Cauchy-Riemann-Equations:

\[
\begin{align*}
∂_x Φ = ∂_y Φ^* \land ∂_y Φ = −∂_x Φ^* \\
\land \text{satisfies the given boundary conditions in the domain } Ω(\vec{z}, t)
\end{align*}
\] \implies Ψ is holomorphic . (C.3)

Examples of such properties include formulas for integration (even for functions that carry singularities [or poles]), and the presence of a harmonic conjugate function. One might wonder if the solutions in 3D are somehow related to the algebra of quaternions. In 2D coordinate systems, it was proven (with results coming from Complex-Analysis), that the existence of a harmonic function, implies the existence of an associated harmonic conjugate function:

The general solution ($Ψ$) to a harmonic equation on the complex plane can be expressed as a sum of one real and one imaginary "sub-functions", which correspond to a pair of harmonic conjugate functions.

In complex coordinates, $z = x + iy$, this is then defined in the following way:

\[
\begin{align*}
∇^2 Ψ(\vec{z}, t) &= 0 \\
\land \text{satisfies the given boundary conditions in the domain } Ω(\vec{z}, t)
\end{align*}
\] (C.4)
All the coefficients in the expansions are complex in general. In other words:

**Convention:**

ordinate systems in 2D and 3D, namely:

variance. Let us now take a look at these solutions (expansions), by comparing them in the most common co-

as the (solutions). Crucially, the functions of these generalized

Equation. Thus, these expansions are at the heart of additional terms, which generalize the

Laplacian are the eigensolutions of the

This equation is commonly referred to as the Poisson Equation. Note that the solutions of this equation

are the eigensolutions of the Laplacian. Understanding these eigenfunctions and the harmonic func-
tions. With particular, as a modern Physicist, one would mean to say beautiful, "magic", harmonious,

harmonic ...
C.3 2D

C.3.1 2D - Cartesian coordinates

In 2D Cartesian coordinates, the Laplacian Equation and the corresponding harmonic expansion take the following form:

\[
\begin{align*}
\nabla^2 f(x, y) &= [\partial_x^2 + 2 \partial_y^2] f(x, y) = 0 \\
\Rightarrow f_h(x, y) &= \sum_{n=-\infty}^{+\infty} [a_n e^{ik_n x} e^{ik_n y} + b_n e^{-ik_n x} e^{ik_n y} + (x \leftrightarrow y)] .
\end{align*}
\]

Conversely, the Helmholtz Equation and the corresponding eigenfunction expansion are:

\[
\begin{align*}
\nabla^2 f(x, y) &= [\partial_x^2 + 2 \partial_y^2] f(x, y) = \Upsilon f(x, y) \\
\Rightarrow f_p(x, y) &= \sum_{n=-\infty}^{+\infty} (a_n r^n + b_n r^{-n}) e^{i \phi} .
\end{align*}
\]

C.3.2 2D - Polar coordinates

In 2D Polar coordinates, the Laplacian Equation and the corresponding harmonic expansion take the following form:

\[
\begin{align*}
\nabla^2 f_h(r, \phi) &= [\partial_r^2 + \frac{1}{r} \partial_r - \frac{1}{r^2} \partial_{\phi}^2] f_h(r, \phi) = 0 \\
\Rightarrow f_h(r, \phi) &= (\alpha \phi + \beta) \log(r) + \sum_{n=-\infty}^{+\infty} a_n r^n e^{i \phi} .
\end{align*}
\]

Conversely, the Helmholtz Equation and the corresponding eigenfunction expansion are:

\[
\begin{align*}
\nabla^2 f_p(r, \phi) &= [\partial_r^2 + \frac{1}{r} \partial_r - \frac{1}{r^2} \partial_{\phi}^2] f_p(r, \phi) = \Upsilon f_p(r, \phi) \\
\Rightarrow f_p(r, \phi) &= (\alpha \phi + \beta) \log(r) + \sum_{n=-\infty}^{+\infty} (a_n r^n + b_n r^{-n}) e^{i \phi} .
\end{align*}
\]

C.4 3D

C.4.1 3D - Cartesian coordinates

In 3D Cartesian coordinates, the Laplacian Equation and the corresponding harmonic expansion take the following form:

\[
\begin{align*}
\nabla^2 f(x, y, z) &= [\partial_x^2 + \partial_y^2 + \partial_z^2] f(x, y, z) = 0 \\
\Rightarrow f_h(x, y, z) &= \sum_{n, m, n, m=\infty} \left\{ [A_{nm} e^{ik_n x} e^{ik_m y} + B_{nm} e^{ik_n z}] e^{i k_n x} + (x \leftrightarrow y \leftrightarrow z) \right\} .
\end{align*}
\]

Conversely, the Helmholtz Equation and the corresponding eigenfunction expansion are:

\[
\begin{align*}
\nabla^2 f(x, y, z) &= [\partial_x^2 + \partial_y^2 + \partial_z^2] f(x, y, z) = \Upsilon f(x, y, z) \\
\Rightarrow f_p(x, y, z) &= \left[ \sum_{n, m, n, m=\infty} \left[ (A_{nm} e^{i n x} + B_{nm} e^{i m y}) e^{\sqrt{k^2 + n^2 + m^2} z} + (x \leftrightarrow y \leftrightarrow z) \right] \right] .
\end{align*}
\]
C.4.2 3D - Cylindrical coordinates

In 3D Cylindrical coordinates, the Laplacian Equation and the corresponding harmonic expansion take the following form:

\[
\begin{aligned}
\nabla^2 f(r, \phi, z) &= \frac{1}{r} \left( \frac{\partial}{\partial r} (r \partial_r) + \frac{\partial^2}{\partial z^2} \right) f(r, \phi, z) = 0 \\
\implies f_p(r, \phi, z) &= \sum_{n, m = -\infty}^{+\infty} \text{Re} \left\{ [A_{nm} I_n(k_{nm} r) + B_{nm} K_n(k_{nm} r)] e^{i(n\phi + k_{nm} z)} \right\}.
\end{aligned}
\]  

(C.16)

where \( I_n(k_{nm} r) \) and \( K_n(k_{nm} r) \) are the modified Bessel functions of the 1st and 2nd kind, respectively. If \( k_{nm} \) is imaginary, they become the Bessel functions of the 1st and 2nd kind. Conversely, the Helmholtz Equation and the corresponding eigenfunction expansion are:

\[
\begin{aligned}
\nabla^2 f(r, \phi, z) &= \frac{1}{r} \left( \frac{\partial}{\partial r} (r \partial_r) + \frac{\partial^2}{\partial z^2} \right) f(r, \phi, z) = \Upsilon f(r, \phi, z) \\
\implies f_p(r, \phi, z) &= \sum_{n, m = -\infty}^{+\infty} \text{Re} \left\{ [A_{nm} I_n(\sqrt{k^2 + m^2} r) + B_{nm} K_n(\sqrt{k^2 + m^2} r)] e^{i(n\phi + k_{nm} z)} \right\}.
\end{aligned}
\]  

(C.17)

C.4.3 3D - Spherical coordinates

In 3D Spherical coordinates, the Laplacian Equation and the corresponding harmonic expansion take the following form:

\[
\begin{aligned}
\nabla^2 f(r, \phi, \theta) &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \partial_r) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \partial_\theta) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} f(r, \phi, \theta) = 0 \\
\implies f(r, \phi, \theta) &= \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} \left\{ A_{nm} R_n^m(r, \phi, \theta) + B_{nm} I_n^m(r, \phi, \theta) \right\}.
\end{aligned}
\]  

(C.18)

where \( R_n^m(r, \phi, \theta) \) and \( I_n^m(r, \phi, \theta) \) are, respectively, the regular and irregular solid harmonics. Note that \( R_n^m(r, \phi, \theta) \propto r^n Y_n^m(\theta, \phi) \) and \( I_n^m(r, \phi, \theta) \propto \frac{Y_n^m(\theta, \phi)}{r^{n+1}} \), where \( Y_n^m(\theta, \phi) \propto e^{im\phi} P_n^m(\cos \theta) \) are the spherical harmonics and thus \( P_n^m(\cos \theta) \) corresponds to associated Legendre Polynomials. Note that \( \propto \) was used instead of \( \sim \), since all these functions are defined with normalizing pre-factors. Conversely, the Helmholtz Equation and the corresponding eigenfunction expansion are:

\[
\begin{aligned}
\nabla^2 f(r, \phi, \theta) &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \partial_r) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \partial_\theta) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} f(r, \phi, \theta) = \Upsilon f(r, \phi, \theta) \\
\implies f(r, \phi, \theta) &= \sum_{n=0}^{+\infty} \sum_{m=-n}^{n} \left\{ A_{nm} r^n + B_{nm} r^{-n-1} \right\} \left\{ Y_n^m(\theta, \phi) \right\}.
\end{aligned}
\]  

(C.19)

C.5 Any coordinate system in D dimensions

An important result to note, is how the generalizations of the harmonic mode expansions (Laplacian), have the same angular dependence, as the eigenfunction expansions (Helmholtz). Mostly, only the radial components of the solutions change from scale-invariant power-laws to Bessel functions of some kind. The reader is encouraged to reflect on this seemingly innocent detail, which will help us argue in favor of quite daring conclusions in the next and last section.

In any arbitrary D-dimensional coordinate system, the Laplacian Equation should still have an expansion as a solution:
The fact that the 3D expansions degenerate smoothly into the 2D expansion is very important. It means that

\[
\nabla^2 f(\vec{q}) \equiv \nabla^2_{qD} f(\vec{q}) = 0 \implies \nabla^2 f(\vec{q}) = \sum_{n_1, n_2, \ldots, n_{D-1} = 0}^{+\infty} \{\ldots\}.
\]

(C.20)

where \(\nabla^2_{qD} f(\vec{q})\) stands for the Laplacian operator in \(D\) dimensions and in the coordinate system \(q\). Likewise, \{\ldots\} represents the variable separated solutions to that Laplacian equation; that is, the respective terms in the infinite harmonic expansion.

\[
\begin{align*}
\nabla^2 f(\vec{q}) &\equiv \nabla^2_{qD} f(\vec{q}) = \Upsilon_{n_1, n_2, \ldots, n_{D-1}} f(\vec{q}) \implies \\
f_{\alpha}(r, \phi, \ldots) &= \sum_{n_1, n_2, \ldots, n_{D-1} = 0}^{+\infty} \{\ldots\}.
\end{align*}
\]

(C.21)

In this section we will compare the solutions to the harmonic equation in 2D-polar coordinates to its generalizations in 3D (cylindrical and spherical coordinates). An important realization is that the solutions to the 3D Laplace Equation (in cylindrical coordinates) degenerate smoothly into the 2-dimensional solution in polar coordinates (ignoring the logarithmic term). In spherical coordinates, the modes do not degenerate into the 2D polar modes exactly, but are qualitatively still very similar. These 3 harmonic expansions are, respectively:

\[
p_{\text{polar}}(r, \phi) = (\alpha \phi + \beta) \log(r) + \sum_{n=-\infty}^{+\infty} r^n \left( A_n \cos(n\phi) + B_n \sin(n\phi) \right),
\]

(C.22)

\[
p_{\text{cyl}}(r, \phi, z) = \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \left[ A_{nm} I_n(k_{nm} r) + B_{nm} K_n(k_{nm} r) \right] e^{i(n\phi + k_{nm} z)},
\]

(C.23)

\[
p_{\text{spher}}(r, \phi, \theta) = \sum_{l=0}^{+\infty} \sum_{m=-l}^{l} A_{lm} R_{lm}^\alpha(r, \phi, \theta) + B_{lm} I_{lm}(r, \phi, \theta).
\]

(C.24)

\[\text{C.6.1 Cylindrical coordinates}\]

Let us now show, how the harmonic expansion in 3D cylindrical can degenerate into the harmonic expansion in 2D polar coordinates. One way to see this, is by noting that the Laplacian operator of the 3D coordinates can degenerate into the operator in 2D polar coordinates. More concretely, when \(\partial_z^2 f\) is small (negligible):

\[
\begin{align*}
\nabla^2_{3D\text{cyl}} f(r, \phi, z) &= \left[ \frac{\partial}{\partial r} (r \partial_r) + \frac{\partial^2}{\partial \phi^2} \right] f(r, \phi, z) \\
\simeq \left[ \frac{\partial}{\partial r} (r \partial_r) + \frac{\partial^2}{\partial \phi^2} \right] f(r, \phi, z) = \nabla^2_{2D\text{polar}} f(r, \phi, z).
\end{align*}
\]

(C.25)

Similarly, the boundary condition should also degenerate into its 2D projection. In the case of surface tension, we have argued previously how given that \(\kappa_{\text{cyl}} = \kappa_{r\phi} + \kappa_{r z}\):

\[
\text{if } \kappa_{r \phi} \gg \kappa_{r z} \implies \kappa_{\text{cyl}} \simeq \kappa_{r \phi} = \kappa_{\text{polar}}.
\]

(C.26)

Recalling that we associated this approximation to be the same as \(|\alpha| < 1\), this brings us to the last argument: in this limit, the 3D mode expansion degenerates into the 2D expansion! If we take \(k_{nm} \to 0\), then

\[
\begin{align*}
\{I_n(k_{nm} r) \to \alpha(k_{nm} r)^n, \frac{\beta}{(k_{nm} r)^n}\}
\end{align*}
\]

(C.27)

If we now set \(\alpha(k_{nm})^n = A^n_{\text{polar}}\) and \(\frac{\beta}{(k_{nm} r)^n} = A^n_{\text{polar}}\), then we get precisely the expansion in polar coordinates. The fact that the 3D expansions degenerate smoothly into the 2D expansion is very important. It means that
the 2D problem is (or at least can be) smoothly connected to a 3D problem. If you will, there is no "phase transition" between them (in the sense of an abrupt qualitative change). This implies that real world 3D problems where one dimension becomes unimportant, can often be understood and modeled as quasi-2D problems. As a consequence, understanding these modes in 2D (polar coordinates) provides us with valuable intuition and insight into the more complicated 3D problems. In other words, the modes in the mentioned 3D coordinate systems contain the 2D modes (qualitatively) and extra modes. This is very important for Hele-Shaw flow, as a base for the pressure corresponds to a base for all possible flow fields. In fact, "star-shaped" oscillations were reported to appear in Hele-Shaw cells, too [13].

More precisely, the cross section of 3D systems in cylindrical coordinates can degenerate exactly into the 2D polar shapes, if the curvature in the other dimension becomes null. This happens for large wavelength, or \( k_{nm} \to 0 \), which corresponds to having a cylinder deformed into the 2D shape at every position in \( z \). This can be imagined as an infinite straight prism with a constant cross-sectional shape (the 2D polar shape).

### C.6.2 Spherical coordinates

If we take \( \theta = \theta_0 = \text{constant} \) in the case of spherical coordinates, then we almost get the polar solution exactly (if \( P_{lm}(\cos \theta_0) \neq 0 \)). There is only an extra pre-factor of \( 2 \) in front of the radial derivative. In the case that \( P_{lm}(\cos \theta_0) = 0 \), we almost get the polar expansion too, if we impose a similar re-definition of the pre-factors, as in the cylindrical case. Another way to think about this, is that the definition of \( \theta \) is arbitrary. Therefore we can always choose it such that

\[
\sin (\theta_0) = 1 \implies P_l[\cos (\theta_0)] = \text{constant} \quad .
\]

However, in spherical coordinates it is not possible to have null curvature in one angular direction, without compromising the integrity of the drop. Thus the 2D projections of the 3D spherical shapes will not be exactly the same as the polar solutions, even though they look very similar. Still, qualitatively, the 2D polar solutions can be thought of as being contained in the 3D spherical solutions. In fact, if we were indeed to set the curvature in \( \theta \) to 0, we would compromise the drops integrity in the sense that we would degenerate into cylindrical coordinates. These in turn do degenerate into the polar solutions exactly.

### C.6.3 Cartesian coordinates

Following the same reasoning as above, it is very easy to confirm, that the harmonic solutions in 3D Cartesian coordinates degenerate very naturally into their 2D counterpart (that is, 2D Cartesian coordinates). In fact, in the case of Cartesian coordinates, this holds from any initial dimension to any final dimension. Acknowledging this, we can appreciate how, in Theory, higher dimensional harmonic modes should always degenerate into their lower dimensional solutions. In a way, this is equivalent to saying: the generalization of a harmonic mode expansion to higher dimensions must contain generalized forms of the initial mode expansion.

### C.7 Scale invariance

Now comes the time to notice, that the harmonic solutions in polar and spherical coordinates are scale invariant functions of the radial scale. Only because they scale as power laws, proportional to \( r^n \), and power laws are self-similar, scale-invariant functions, were we able to find self-similar pressure and flow-profiles in the linear solution. Notwithstanding, this heavily depends on the boundary conditions, which can introduce scale variance! Likewise, non-linear terms and generalizations of the Laplacian equation, might inevitably destroy the fractal beauty of the harmonic mode expansion.

Note now, that we have left the logarithmic term out of the discussion. That is because it is clearly not a power law, nor is it a scale invariant function. Nevertheless, it can still produce scale invariant motion, as its derivative is a power law: \( \partial_r \log(r) = \frac{1}{r} = r^{-1} \). This is the case of an ideal vortex in 2D (or sink/source), where the pressure and the velocity field are given by:

\[
\begin{align*}
\begin{cases}
    p(r, \phi) = c \log(r) & , \quad c \in \mathbb{C} \\
    \vec{u}(r, \phi) = \frac{\Gamma}{2\pi r} (\text{Re}\{c\} \vec{e}_r + \text{Im}\{c\} \vec{e}_\phi)
\end{cases}
\end{align*}
\]

Thus, the only mode that is not scale-invariant in both the polar and cylindrical harmonic expansions. It is, however, the only term that can achieve a \( r^{-1} \) law in its derivative. Without that power law, we would not
be able to describe 2D vortex-, sink- and source-flows; One could think that it is thus required even, and as a matter of fact it is quite naturally included, as it solves the 2D Laplace equation. The cylindrical coordinates are a quite interesting case: due to the Bessel functions, the potential is not scale invariant in the radial dimension. This is essentially due to the fact that there is a distinction between length-scales, $\varepsilon_r$ and $\varepsilon_z$. As soon as there is a distinction, the symmetry is broken, and so is the scale-invariance. Nonetheless, the Bessel functions of the 2nd kind degenerate into $r^n$, and the modified Bessel functions of the 2nd kind degenerate into $r^{-n}$, respectively. Thus the harmonic expansions behave quite self-similarly for $|ka| \ll 1$. This is equivalent to saying, that the changes in $\varepsilon_r$ are much more important than $\varepsilon_z$.

Last but surely not least, we would like to underline, how "easily" this whole formalism could be generalized to arbitrary dimensions, ND. In higher dimensions, one can only wonder, whether the correspondence to the complex plane (which in turn gives rise to harmonic functions and harmonic conjugate functions) generalizes to more abstract forms of complex algebras, like quaternions, octonions and, potentially, to "N-tonions".

### C.8 Boundary conditions

There are many forms of boundary conditions. Infinitely many could be constructed mathematically, but as it turns out, there is a set of very few, simple and recurring boundary conditions in physics. The arguably most distinguished ones are the so called Drichlet- (or closed) and Neumann- boundary conditions (or open).

As mentioned in the previous section, scale variance can be (and usually is) introduced in the mathematical description of a system, through its boundary conditions. In fact, very often, this is just because the boundary conditions "force" us to admit the system to have a certain finite size. This gives the system a specific, finite and measurable length-scale. This scale, $l_0$ now can be thought of as having to play a role in the dimensional analysis of the problem under study (consider the Buckingham-II-Theorem). Grasping this thought already permits one to see, how the seemingly scale invariant harmonic expansion will often produce scale variant results. A neat and previously discussed example hereof is the surface tension of a fluid. Surface tension is effectively a boundary condition, since it expresses a pressure drop across a surface that separates 2 fluids (as described by the Young-Laplace equation). To be more precise, it is an extra term to be considered in Neumann-boundary conditions. It can be shown, that surface tension is doomed to be a weak force for length-scales above the capillary length (3mm for water). Given that it acts with different degrees of importance, depending on the length-scale of the system, $l_0$, it thus expresses scale variant boundary conditions. Notwithstanding, for small amplitudes most boundary conditions will be linear (if the system is stable) and produce the scale invariant harmonic modes around the equilibrium shape (circle/sphere/cylinder/...).

### C.9 Natural time scales

As we have just seen, some boundary conditions introduce scale variance, fixing a length-scale of a particular physical system in the process.

The dynamics of physical systems, which quite often relate to the Laplacian operator and its self-similar harmonic expansions, can generally be categorized into either stability or instability (depending on whether the initial configuration is stable or unstable). Both of those scenarios are generally characterized by a typical time-scale, namely (and respectively) an oscillation period $T = \frac{2\pi}{\omega}$, or a relaxation time $\tau_{\text{rel}}$.

Note, however, that the way both of these depend on the length-scale (boundary condition) is generally similar. For instance, the oscillation frequency of a drop due to surface tension, and the viscous diffusion (dissipation) of momentum scale, respectively, as:

$$\omega \propto l_0^{-\frac{3}{2}} \quad \Rightarrow \quad T \propto \quad \tau_{\text{diff}} \propto \frac{l_0^{\frac{3}{2}}}{l_0} .$$

(C.30)

Following this line of reason, it seems that the dynamic time scale(s) of a particular system are determined by the length-scales and the boundary conditions hereof. This happens as a particular side-product of the scale variance (generally boundary conditions) of our system. Take the harmonic modes of a vibrating string attached to 2 walls, for instance:

$$f_n = n \frac{c}{\lambda} \quad \Rightarrow \quad T_n \propto l_0 .$$

(C.31)

As one can see, a natural time-scale of a concrete physical system, under certain dynamics is subordinated by its length-scale(s) and its dynamics. Furthermore, the natural dynamical time scales seem to increase with increasing length-scale. In other words, most seem to obey $t \propto l_0^\alpha$, with $\alpha \geq 0 \in \mathbb{Q}$ as a tendency.
C.10 “Subverses”

This notion can be abstracted further: has the reader ever wondered, why things at small scales seem to express
at intrinsically fast dynamics, whereas big-scaled systems seem to evolve at very slow rates? We can subdivide
different length-scales using a logarithmic scale, by defining “scale classes”, for example:

\[
\begin{align*}
\text{Nuclear scale} & \propto 10^{-15} \text{m} = 1 \text{fm} \\
\text{Atomic scale} & \propto 10^{-10} \text{m} = 1 \text{Å} \\
\text{Molecular scale} & \propto 10^{-8} \text{m} = 10 \text{nm} \\
\text{Cellular scale} & \propto 10^{-6} \mu \text{m} \\
\text{Human scale} & \propto 10^1 \text{m} = 1 \text{m} \\
\text{Environmental scale} & \propto 10^3 \text{m} = 1 \text{km} \\
\text{Planetary scale} & \propto 10^6 \text{Mm} \\
\text{Galactical scale} & > 10^{12} \text{Tm}
\end{align*}
\]

(C.32)

Note that more or less “scale classes” could be defined; both in between these, and at both ends. Or they
could simply be defined in a completely different way. The point here is, that comparing 2 of these “scale
classes”, as for instance the Cellular scale opposed to the Human scale, they seem like 2 biological Universes
of their own. In particular, it seems that the Cellular “subverse” works intrinsically much faster than the Human
“subverse”. This can now be put into perspective, by comparing the time scales of typical dynamics at both
scales. For example, compare the behaviour of a star-shaped Leidenfrost drop oscillating and a cell, subject to
a similar surface perturbation:

\[
\begin{align*}
\text{surface tension} \left( \frac{T_{\text{Human}}}{T_{\text{Cell}}} \right) & \propto \left( \frac{1}{10^{-6}} \right)^2 = 10^9 \\
\text{diffusion} \left( \frac{\tau_{\text{Human}}}{\tau_{\text{Cell}}} \right) & \propto \left( \frac{1}{10^{-6}} \right)^2 = 10^{12}
\end{align*}
\]

(C.33)

We can understand the similar (polygonal) structures emerging at different scales of the Universe to be a product
of linear boundary conditions with (pure) harmonic modes. The diversity of completely different and coexisting Subverses, can then be understood as the breaking of the self-similarity of these fractal (simplified)
laws. This loss of self-similarity is owed to the introduction of length-scale variance, which is inevitably entangled
with dynamical time-scale variance. Setting a length-scale, will determine the typical time of typical
dynamics of the system. There is an alternate way to understanding this: let us define an adimensional number
inspired in the Deborah Number, De:

\[
De \equiv \frac{\tau_{\text{relaxation}}}{t_{\text{observation}}},
\]

(C.34)

where the relaxation time, \( \tau \), symbolizes the natural dynamical time-scale (or typical reaction time) of the
“subverse”. It could, for instance, represent viscous dissipation, or typical vibrations or waves of said “subverse”. Just any type of typical intrinsic time scale. Then if you would want to “live” in a particular “subverse”, that is, if you would like to perceive that “subverse” for what it is in its natural time-scale, then arguably, you
should be able to observe (react) at a scale \( De \simeq 1 \). Since this is not possible for distant “scale classes”, they
seem to be distant Universes of their own, living around you. For instance, the Geological formations seem static to us, or the growth of a mountain, or the formation of a galaxy. But if we made a time lapse, it would look like a very dynamic process, resembling a flow of some sort. Likewise, taking very fast snapshots of our experiments, we were able to produce the pictures in Figure 1.19 to Figure 2.11 (with the shutter speed of the order of pinch-off time scales, or smaller).

Understanding the life and dynamical richness of each Subverse thus corresponds to observing at a time scale,
which matches the typical dynamical scale of a physical system (forcing \( De \simeq 1 \)). In a way, this seems to be a congruent hypothesis to account for the diverse, yet subtly similar dynamics found throughout the Universe.

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